APEX CALCULUS II

Version 4.0

Improper Integral

$$\int_{-\infty}^{+\infty} \frac{x\,dx}{1+x^2} \qquad u = 1 + x^2,\ du = 2x\,dx$$

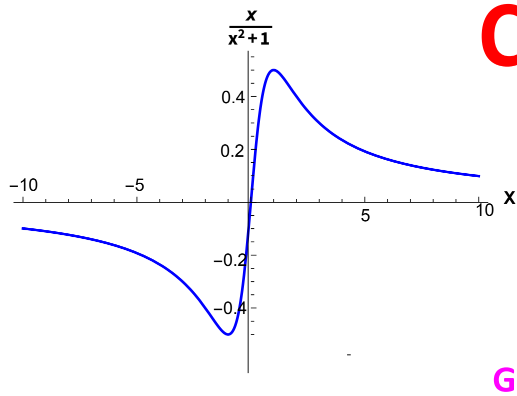
$$= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{2x\,dx}{1+x^2}$$

$$= \frac{1}{2}\ln(1+x^2)\Big|_{-\infty}^{+\infty}$$

$$= \{\infty - \infty\},$$

indeterminate form

Does Not Exist



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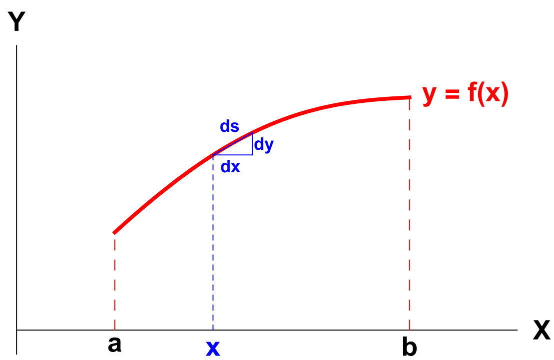
Arc Length

What is the arc length, the length of the curve, of $y = f(x)$, $a \le x \le b$?



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Theorem Let $y = f(x)$ be differentiable for $a \le x \le b$. Then its arc length on that interval is

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx.$$

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Derivation

f is differentiable on the interval. So f is differentiable at x and therefore locally linear or equivalently 'asymptotically straight' there. Therefore

$$ds^2 \approx dx^2 + dy^2 \qquad \text{Theorem of Pythagoras}$$

$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right] dx^2$$

$$ds \approx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\Longrightarrow \quad s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx.$$

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Apex Infinitesimal Calculus, Volume II

Third Preliminary Version

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CONTRIBUTIONS

Lyryx Calculus 10.1, 2, 3

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Preface

Infinitesimal Calculus, Volume II

In Volume I we set up, in detail, the basics of hyperreal calculus. The payoff was being able to do a thorough, elementary and intuitive development of calculus. Proofs of the all basic operational formulas and theorems were accomplished as well as more difficult foundational theorems such as the **Extreme Value Theorem** and the **Riemann Integrability of Continuous Functions over a Closed Interval**.

In this Volume II, early/late transcendentals, we continue the theoretical development - especially applications of integration - using hyperreal analysis. Historically, the fundamental starting point of calculus was the ***differential***. This is because the basic laws of science and geometric are often simpler to discover over an infinitely short interval of space or time. Now, with the basics of elementary applications better understood, it makes sense to start where students have better mathematical strengths.

To find the rate of change of Q with respect to t, the ***derivative***, you divide the differential by dt. To find the change in Q from time t_1 to t_2, the ***integral*** of f(t) over the interval, you sum the infinitesimal f(t) dt's from t_1 to t_2 and find the closest real number to this sum.

$$\frac{dQ}{dt} = f(t)$$

$$\uparrow$$

$$dQ = f(t)\,dt \qquad \textbf{the differential}$$

$$\downarrow$$

$$\Delta Q = \int_{t_1}^{t_2} f(t)\,dt$$

The greatest benefit of infinitesimal methods is providing a reliable, intuitive and foolproof guide to setting up applications of integration.

The differential in modern textbooks is a real number and is relegated to a method of approximating a function near a point. It is also used somewhat dishonestly and without further explanation in order to be able to employ powerful infinitesimal techniques such as the method of change of variable in integration and separation of variables in differential equations. The symbol $\frac{dy}{dx}$ now is a fraction and needs no awkward discussion and the chain rule is obvious and proves itself (check it out in Volume I.

The great tools of infinitesimal calculus are **infinitesimals**, which facilitates the ultra-precise calculations used to do calculus and the **equivalence relation ≈** which enables the rapid comparison and simplification of hyperreal expressions.

Later misuse of the key comparison symbols **=, ≈** and **≈>** is not a big problem because their outputs tend to differ only by an infinitesimal. ≐ is the approximate equality for real numbers.

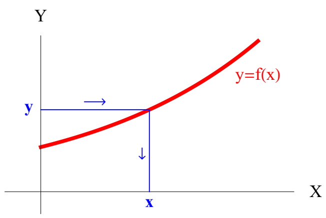
Chapter 6 Advanced Transcendental Functions

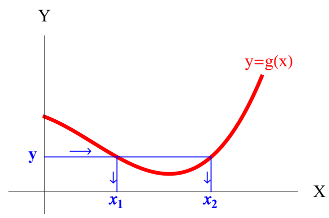
6.1 Inverse Functions Theory - a Review

The main purpose of the inverse function is to solve for the independent variable x in y = f(x):

$$\boxed{y = f(x) \iff x = f^{-1}(y)}$$

I. When does f have an inverse? Answer: if for each y in the range of f, there is exactly one x.





Each y, one x. Some y's, more than one x.

We say that f is ***one-to-one*** or ***invertible*** if for each y, there is exactly one x (or equivalently f passes the *Horizontal Line Test*.) Note that increasing functions are invertible; so are decreasing functions.

II. How to find the formula for f^{-1}

y = f(x)

Solve for x:

$x = f^{-1}(y)$

III. The inverse function $y = f^{-1}(x)$ We prefer as usual, in working with or studying f^{-1} to call the dependent variable y. So interchanging x and y:

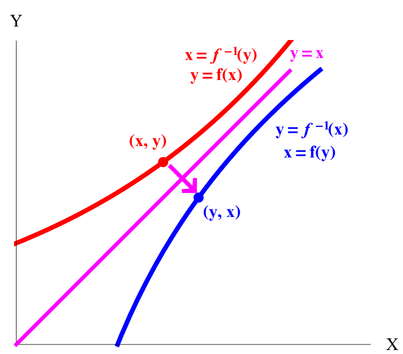
$y = f^{-1}(x)$

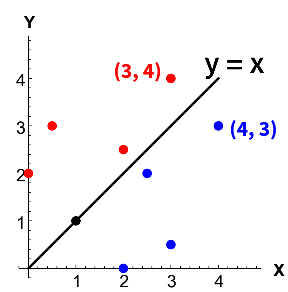
(In applications, f^{-1} is usually of interest only when it cannot be found exactly! f^{-1} is then found numerically by computer.) Note that y = f(x) and $y = f^{-1}(x)$ have different graphs and therefore are different functions.

IV. How to graph $y = f^{-1}(x)$ Since we interchanged x and y to obtain the graph of $y = f^{-1}(x)$ from that of y = f(x):

Reflect the graph of y = f(x) across the line y = x.

Note which equations have the same graph.





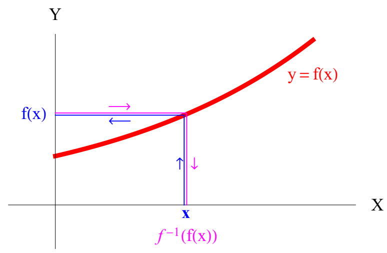
How it works

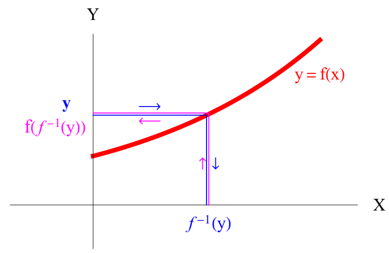
The point (a, b) reflected across the line y = x is the point (b, a).

V. Inverse Function Identities

$$f^{-1}(f(x)) = x$$
$$f\left(f^{-1}(y)\right) = y$$

These identities are verified by following the arrows in the graphs below. Start at x in the left graph and at y in the right graph.





VI. Application: Universal Equation Solver

Solve:

$$f(x) = c$$

$$f^{-1}(f(x)) = f^{-1}(c)$$ Taking f^{-1} of both sides

The solution:

$$\boxed{x = f^{-1}(c)}.$$ Inverse function identity

VII. Calculus of a inverse function

$$y = f^{-1}(x) \iff x = f(y)$$

$$1 = f'(y)\,\frac{dy}{dx}$$ Differentiating implicitly; Chain Rule

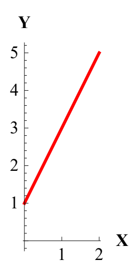
$$\frac{dy}{dx} = \frac{1}{f'(y)}$$

or

$$\boxed{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}.$$ Live math.

Easy Example **y = f(x) = 2x + 1**

I.



Passes the Horizontal Line Test. So $f^{-1}(x)$ exists.

II. Find the formula for f^{-1} ?

$y = f(x) = 2x + 1$

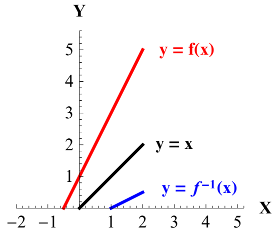
Solve for x : $x = f^{-1}(y)$

$x = f^{-1}(y) = \frac{1}{2}y - \frac{1}{2}.$

III. The inverse function. Interchange x and y.

$y = f^{-1}(x) = \frac{1}{2}x - \frac{1}{2}.$

IV. Its graph.



V. Inverse function Identities. Let us check.

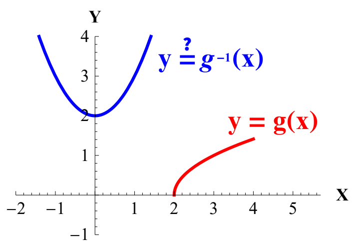
$$f^{-1}(f(x)) = \frac{1}{2}f(x) - \frac{1}{2} = \frac{1}{2}(2x+1) - \frac{1}{2} = x.$$

$$f(f^{-1}(x)) = 2f^{-1}(x) + 1 = 2\left(\frac{1}{2}x - \frac{1}{2}\right) + 1 = x.$$

Properties VI and VII are left as exercises. They are important for more difficult examples.

Harder Example $y = g(x) = \sqrt{x-2}$

If you solve for x and then interchange x and y, you get $y = g^{-1}(x) \overset{?}{=} x^2 + 2$. Lets see.

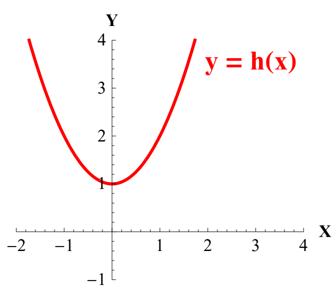


Recall that squaring an equation can give spurious solutions. Clearly, reflecting y = g(x) about the line y = x, the correct inverse is

$$\mathbf{y = g^{-1}(x) = x^2 + 2, x \geq 0}.$$

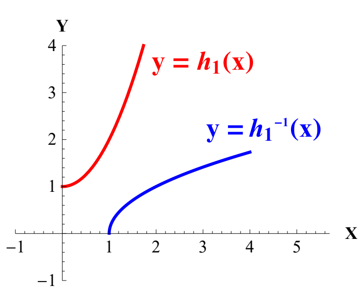
Another Example with a problem $y = h(x) = x^2 + 1$

If you solve for x and then interchange x and y, you get $y = h^{-1}(x) \overset{??}{=} \pm\sqrt{x-1}$. Let's see.

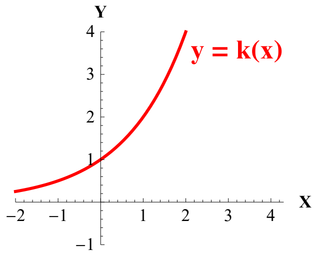


The problem here is that the function y = h(x) is not one-to-one. So it does not have an inverse. Nevertheless, it will be useful at times to do the best we can in finding a related inverse. What we do is restrict the domain to x ≥ 0; what remains is one-to-one. (We could, of course, made the domain choice x ≤ 0. In many advanced applications, there are good reasons for one choice over another.) Conclusion:

$$y = h_1(x) = x^2 + 1, x \geq 0$$
$$y = h_1^{-1}(x) = \sqrt{x-1}$$



A Very Difficult Problem $y = k(x) = 2^x$



$y = k(x)$ is a one-to-one function. So its inverse exists. However, there is no elementary way to solve for x in terms of y.
What one does in such cases is to give the inverse function a name and let a computer program figure out the value of y for each x. ***Inverse functions are most important when you can't solve for x explicitly!***

In this case, as you may remember from high school math:

$y = k(x) = 2^x$

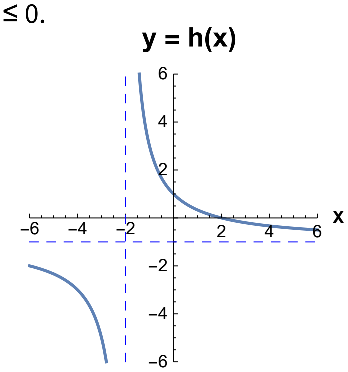
$y = k^{-1}(x) = \log_2(x)$, " log, base 2, of x."

Every one-to-one function has an inverse which automatically satisfies properties I to VI and VII also if the function is suitably differentiable. This knowledge gives you much information about the inverse function which it inherits from the original function. This is very helpful in that beginners often find inverse functions particularly difficult. In addition, any properties peculiar to to the original function also translate into properties of the inverse function. Keep this in mind as we continue through this chapter.

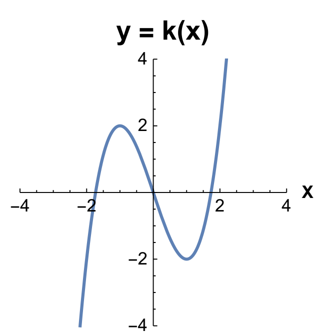
Exercises 6.1

In each problem, do steps I to V. Check by graphing or calculating. Semi-memorize the seven steps.

1. $y = f(x) = 1 - x^2$, $0 \le x \le 1$.
2. $y = g(x) = \sqrt{1 - x^2}$, $-1 \le x \le 0$.
3. $y = h(x) = \frac{2-x}{2+x}$. Hint:



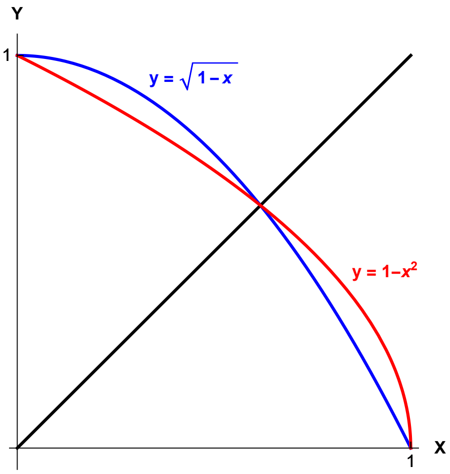
4. $y = k(x) = x^3 - 3x$. Find one partial inverse.
   You can use a CAS or Wolfram Alpha to solve for x. Hint:



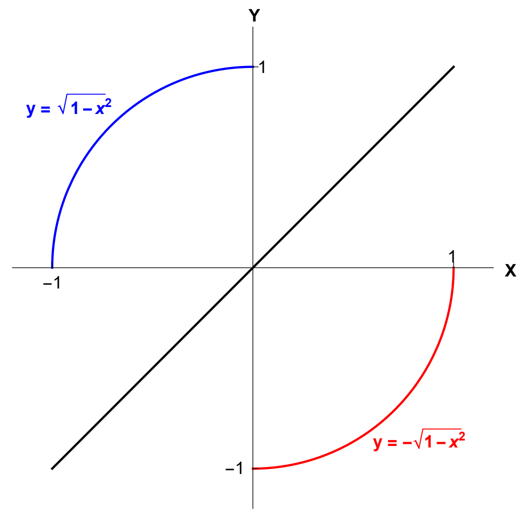
5. $y = 3^x$

Solutions

1.

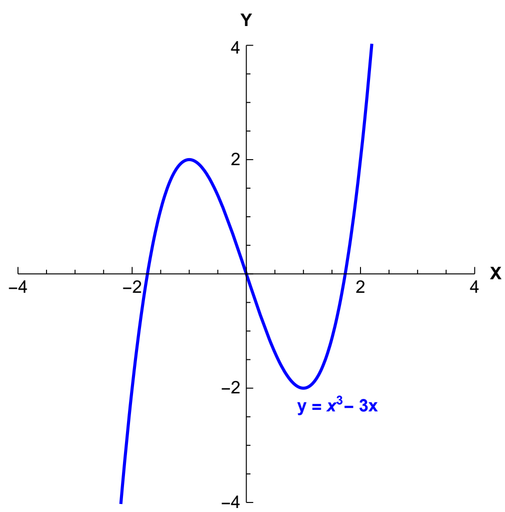


2.

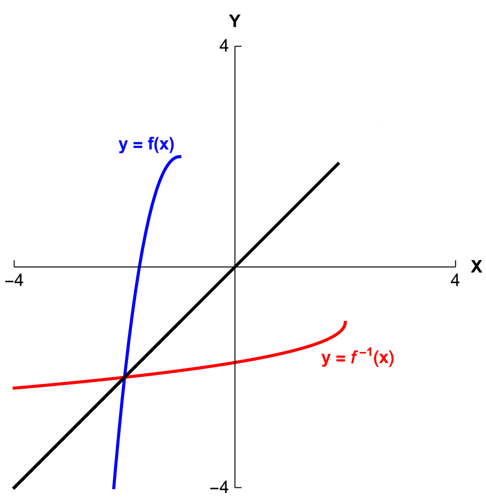


#4. $y = f(x) = x^3 - 3x$. Solve for x and interchange x and y. We will get the leftmost of the three possible inverses using a CAS:

$$y = f^{-1}(x) = -\frac{2^{1/3}}{\left(-x+\sqrt{-4+x^2}\right)^{1/3}} - \frac{\left(-x+\sqrt{-4+x^2}\right)^{1/3}}{2^{1/3}}.$$



Graphing



6.2 Exponential Functions Review. The Natural Exponential Function

Review of Exponents

b^x ← exponent
base ↗

Definition of Exponents For n a natural number:

$b^1 = b$
$b^2 = b \cdot b$
$b^3 = b \cdot b \cdot b$
$\vdots$
$b^n = b \cdot b \cdot b \cdots b \cdot b$, n factors of b.

Properties of Exponents (m, n natural numbers)

1. $b^m b^n = b^{m+n}$ Example: $b^2 \cdot b^3 = (b \cdot b)(b \cdot b \cdot b) = b^{2+3}$
2. $\frac{b^m}{b^n} = b^{m-n}$ Example: $\frac{b^5}{b^3} = \frac{b \cdot b \cdot b \cdot b \cdot b}{b \cdot b \cdot b} = b^{5-3}$, cancelling 3 common factors
3. $(b^m)^n = b^{m \cdot n}$ Example: $(b^2)^3 = (b \cdot b)(b \cdot b)(b \cdot b) = b^{3 \cdot 2} = b^{2 \cdot 3}$, two groups of three

We would like to define b^x for other real numbers x. We do this in such a way that the ***Properties of Exponents*** hold.

Definition $b^0 = 1$ because, for example, $b^0 = b^{3-3} = \frac{b^3}{b^3} = 1$ (Property 2)

Definition $b^{-n} = \frac{1}{b^n}$ because $b^{-n} = b^{0-n} = \frac{b^0}{b^n} = \frac{1}{b^n}$

What about fractional powers? Consider $b^{1/2}$: $b^{1/2} \cdot b^{1/2} = (b^{1/2})^2 = b^1 = b \implies b^{1/2} = \sqrt{b}$

Definitions:

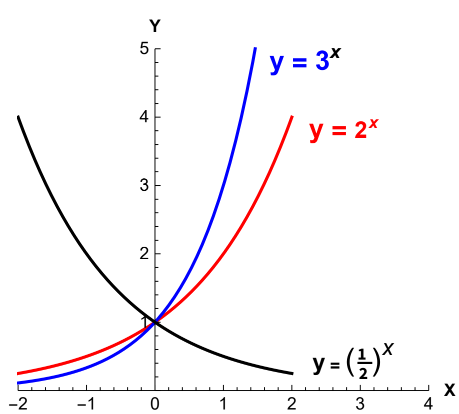
$b^{1/n} = \sqrt[n]{b}$

$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$

Recall that if n is an even integer, b must be non-negative for m odd.

Definition *Exponential Function with base b*

$y = b^x$



Domain: all x
Range: $0 < y < +\infty$

$0 < b < 1 \Rightarrow$ decreasing function
$b > 1 \Rightarrow$ increasing function

Note: Exponential functions are not defined for bases $b < 0$.

Do not allow because, for example, because $y = (-2)^x$ is not a real number if $x = \frac{1}{2}$.

Choosing a base for calculus

Which base b has the 'nicest' derivative?

$$\frac{d}{dx}(b^x) = \frac{b^{x+dx} - b^x}{dx} = \frac{b^{dx} - 1}{dx}\, b^x$$

Answer: if $\frac{b^{dx} - 1}{dx} \approx> 1$. Exploring this using limit approximations:

if $\frac{b^h - 1}{h} \longrightarrow 1$ as $h \longrightarrow 0$

or $b^h - 1 \longrightarrow h$ as $h \longrightarrow 0$

or $b^h \longrightarrow 1 + h$ as $h \longrightarrow 0$

or $b \longrightarrow (1+h)^{1/h}$ as $h \longrightarrow 0$

Let us calculate (note that these calculations are elementary but tedious):

h	$(1+h)^{1/h}$	
1	$(1+1)^1$	$= 2$
0.1	$(1.1)^{10}$	$= 2.5937424601$
0.01	$(1.01)^{100}$	$\doteq 2.748138294$
0.001	$(1.001)^{1000}$	$\doteq 2.716923$
0.0001	$(1.0001)^{10000}$	$\doteq 2.7181458$
↓		↓
0		$2.718\cdots$

Why didn't we do an exact derivation or proof? Unfortunately for advanced functions there is often no elementary algebra that can do the job. So we resort to a numerical procedure.

> So the base which has the nicest derivative is
>
> $e = 2.718\cdots$
>
> called ***Euler's Constant***. Leonard Euler, 1707 to 1783, was the world's most prolific calculus mathematician.

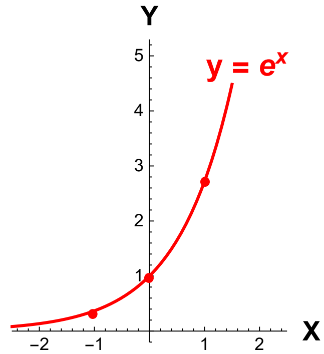
Euler is pronounced 'Oiler'

Definition The ***natural exponential function*** is

$$\mathbf{y = e^x,\ e = 2.718281828459045\cdots}$$

Memorize this. Easier than π. Amaze your friends.

$$\frac{d}{dx}(e^x) = e^x \qquad \int e^x dx = e^x + C$$



Tables of values for graphing by hand

x	e^x
-2	0.14
-1	0.37
0	1
1	2.7
2	7.4

Financial Application of The Natural Exponential Function

You learned in high school about compound interest. Compounding means that after a period of time, the interest earned is added to the initial amount and the new larger amount continues earning interest. The future value F of a present value P at a yearly interest rate r compounded n times yearly (a high school formula) is

$$F = P\left(1 + \frac{r}{n}\right)^{nt}.$$

What if compounding is done continuously ($n \to +\infty$)?

$$\begin{aligned} F &= \lim_{n\to+\infty} P\left(1 + \frac{r}{n}\right)^{nt} \\ &= \lim_{n\to+\infty} P\left(1 + \frac{r}{n}\right)^{\frac{n}{r}\cdot rt} \\ &= \lim_{h\to 0} P\left[(1+h)^{\frac{1}{h}}\right]^{rt} \qquad \text{Letting } h = \frac{r}{n} \\ &= P\,e^{rt} \qquad \textbf{Note } \text{By the definition of } e\text{: } e = \lim_{h\to 0}(1+h)^{\frac{1}{h}} \end{aligned}$$

$$\boxed{F = Pe^{rt}}$$

Example You take the amount of money, \$200,000, your mom gave you at age 20 to go to university but instead invested it at a rate of 10% compounded continuously. Will you be able to retire from your McRonald's job at age 65 and live well?

$$\begin{aligned} F &= 200000\, e^{0.1(65-20)} \\ &= 200000\, e^{4.5} \\ &= \$\,18{,}000{,}000 \end{aligned}$$

Yes! (but watch out for inflation)

Exercises 6.2

1. On the same graph, plot by hand $y = e^x$, $y = e^{2x}$ and $y = e^{x/2}$.
2. On the graph, plot by hand $y = e^{-x}$ and $y = e^{-2x}$.
3. On the same graph, plot by hand $y = e^{x^2}$ and $y = e^{-x^2}$.
4. On the same graph, plot by hand $y = e^x$ and $y = e^{(x-2)}$
5. Prove that $x^{1/3} = \sqrt[3]{x}$. Hint: cube the left hand side.
6. $\frac{d}{dx}(e^{3x}) =$ $\qquad$ $\frac{d}{dx}\left(e^{x^2}\right) =$

   $\frac{d}{dx}\left(e^{\sin x}\right) =$ $\qquad$ $\frac{d}{dx}\left(e^{x^2 \cos x}\right) =$

   $\frac{d}{dx}\left(\left(x^2 + 7\right) e^{\sin x}\right) =$ $\qquad$ $\frac{d}{dx}\left(\frac{e^{x^2}}{x+1}\right) =$

   $\frac{d}{dx}(\tan(e^{3x})) =$ $\qquad$ $\frac{d}{dx}(e^{\tan x}) =$
7. $\int e^t \, dt =$ $\qquad$ $\int e^t \cos(e^t) dt =$

   $\int x e^{x^2} \, dx$ $\qquad$ $\int e^x e^{e^x} dx$

8. Find the area under the curve $y = e^x$ for $0 \le x \le 1$. Illustrate with a graph.
9. You decided to go to university anyway. But you decided to save the amount you budgeted for lunches for the four years, \$15,000, and invest it at 10% for 45 years? Should you get a job with retirement benefits?
10. Derive the formula $F = P\left(1 + \frac{r}{n}\right)^{nt}$. Start with the simple interest formula $F = P(1 + rt)$. Derive this last formula first.
11. A certain bacterium divides every hour. How many will there be after 24 hours.
    a. Use the exact formula.
    b. Compare with the continuous compounding formula. $F = Pe^{rt}$.

Solutions 6.2

6. $e^{3x} \cdot 3$ $\qquad$ $e^{x^2} \cdot 2x$

   $e^{\sin x} \cdot \cos x$ $\qquad$ $e^{x^2 \cos x}(2x \cos x - x^2 \sin x)$

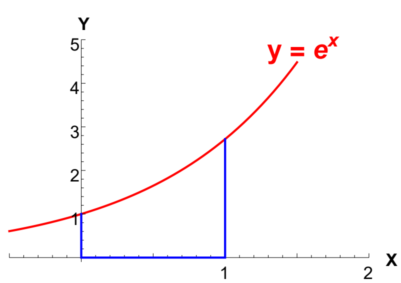
   $2x\, e^{\sin x} + (x^2 + 7)\, e^{\sin x} \cdot \cos x$ $\qquad$ $\frac{e^{x^2} 2x(x+1) - e^{x^2}}{(x+1)^2}$

   $\sec^2(e^{3x}) e^{3x} \cdot 3$ $\qquad$ $e^{\tan x} \sec^2 x$

7. $e^t + C$ $\qquad$ $\sin e^t + C$

   $1/2\, e^{x^2} + C$ $\qquad$ $e^{e^x} + C$

8. e - 1



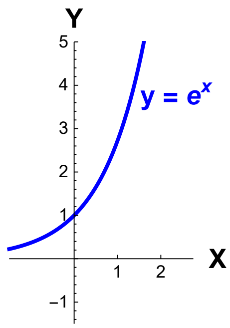
11 a. $N = 1\left(1 + \frac{1}{1}\right)^{1 \cdot 24} = 16\,777\,216$

b. $N \overset{?}{=} 1\, e^{1 \cdot 24} = 2.64891 \times 10^{10}$ This is vastly greater than the correct answer 16,777,216.
Here the bacteria start growing immediately, whatever this means; they do not wait an hour.

6.3 The Natural Logarithmic Function. Other Bases

The inverse of the natural exponential function is the natural logarithmic function. We will follow the steps of section 6.1 to discover its inverse function properties.

I. Does $y = f(x) = e^x$ have an inverse so that $y = f(x) = e^x \Longleftrightarrow x = f^{-1}(y)$?



Yes, it satisfies the horizontal line test.

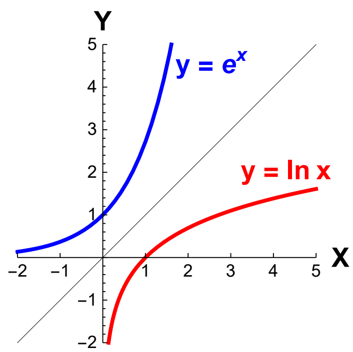
II. Find its inverse function by solving
$y = f(x) = e^x$ for x.

This cannot be done by elementary algebra. So we pretend we can and write
$x = f^{-1}(y) = \ln(y)$ **Note** "logarithms are exponents"

where $\ln y$ is called the natural logarithm of y. ln is the abbreviation of its Latin name *Logarithimus Naturalis*. It is pronounced 'ell-n x' or 'lon (rhymes with Ron) x' or in advanced work is written 'log x'.

III. The natural logarithm function. $x \leftrightarrow y$.
$y = \ln x$

IV. The graph of $y = \ln x$



Note **The values of *named functions* such as the *natural logarithmic function* typically are calculated numerically by computer and which you can then freely use in any application.**

V. The Inverse Function Identities

$\ln(e^x) = x$ 'easy logs'

$e^{\ln x} = x$ 'exponentiation'

Example 2 in base e form is
$2 = e^{\ln 2} \doteq e^{0.693}$

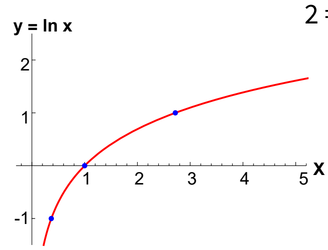
Examples

$\ln \frac{1}{e} = \ln e^{-1} = -1$

$\ln 1 = \ln e^0 = 0$

$\ln e = \ln e^1 = 1$

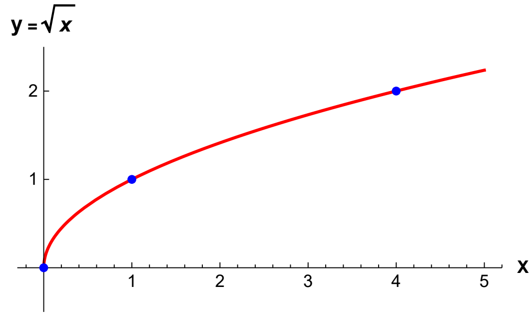
$\ln e^2 = 2$



These values are useful for graphing $y = \ln x$ by hand.

Compare with the case of ***easy roots***, useful for the same reason.

$$\sqrt{0} = \sqrt{0^2} = 0$$
$$\sqrt{1} = \sqrt{1^2} = 1$$
$$\sqrt{4} = \sqrt{2^2} = 2$$
$$\vdots$$



VI. Solving Exponential Equations

Example Solve

$e^{2x} = 7$	
$\ln e^{2x} = \ln 7$	Take the log of both sides
$2x = \ln 7$	Easy logs
$x = \frac{1}{2}\ln 7$	
$\doteq 0.973$	Calculator.

VII. Calculus

$y = \ln x \iff x = e^y$

$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}.$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

$$\boxed{\int \frac{du}{u} = \ln|u| + C}$$

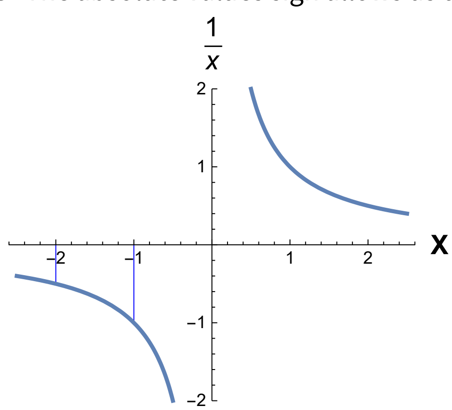
Verify that the absolute value sign in the integral is appropriate.

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}\begin{cases} \ln(-x) & x < 0 \\ \ln(x) & x > 0 \end{cases} = \begin{cases} \frac{d}{dx}\ln(-x) & x < 0 \\ \frac{d}{dx}\ln(x) & x > 0 \end{cases} = \begin{cases} \frac{-1}{-x} & x < 0 \\ \frac{1}{x} & x > 0 \end{cases} = \frac{1}{x},\ x \neq 0.$$

Example

$\frac{d}{dx}(\ln(\sin x)) = \frac{1}{\sin x}\cdot\cos x = \cot x$ Chain Rule

Example The absolute values sign allows us to find the ‘area’ below from $x = -2$ to $x = -1$.



$$\int_{-2}^{-1} \frac{dx}{x} = \ln|x|\Big|_{-2}^{-1}$$
$$= \ln|-1| - \ln|-2|$$
$$= 0 - \ln 2$$
$$= -\ln 2$$
$$\doteq -0.693$$

Further Properties of Logarithms

The Properties of Exponents

1. $e^x e^y = e^{x+y}$
2. $\frac{e^x}{e^y} = e^{x-y}$
3. $(e^x)^y = e^{xy}$

translate directly into

Properties of Logarithms

1. $\ln(xy) = \ln x + \ln y$
2. $\ln \frac{x}{y} = \ln x - \ln y$
3. $\ln x^y = y \ln x$

Proof of 1 $\ln(xy) = \ln x + \ln y$

$\ln(xy)$

$= \ln\left(e^{\ln x} e^{\ln y}\right)$ exponentiation

$= \ln\left(e^{\ln x + \ln y}\right)$ property 1 of exponents

$= \ln x + \ln y$ easy logs

Other Bases

$$\frac{d}{dx}(b^x) = b^x \ln b \qquad \int b^u \, du = \frac{b^u}{\ln b} + C$$

$$\frac{d}{dx}(\log x) = \frac{1}{x \ln b}$$

Proof

$y = b^x \iff \ln y = x \ln b$

$\frac{1}{y}\frac{dy}{dx} = \ln b$

$\frac{dy}{dx} = y \ln b$

$= b^x \ln b$

Proof

$y = \log_b x \iff x = b^y$

$1 = b^y \frac{dy}{dx} \ln b$

$\frac{dy}{dx} = \frac{1}{b^y \ln b}$

$= \frac{1}{x \ln b}$

or do by general property VII of inverse functions.

Logarithmic Differentiation

There is one more general derivative formula we need. We do not yet know how to differentiate

y = $f(x)^{g(x)}$.

Method: take ln of both sides.

$$\ln y = \ln(f(x)^{g(x)})$$

$$= g(x)\ln(f(x)) \qquad \text{Property 3 of logarithms}$$

$$\frac{1}{y}\frac{dy}{dx} = g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)} \qquad \text{Implicit differentiation}$$

or

$$\frac{dy}{dx} = y\left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right)$$

$$= f(x)^{g(x)}\left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right)$$

NOTE Do not memorize this formula.
Go through this process for each example.

Example Graph $y = x^x$ The natural domain of this function is $x > 0$.

x	x^x
1	$1^1 = 1$
2	$2^2 = 4$

$\lim_{x\to 0} x^x = 1$ You will learn how to do this exactly in Section 6 or now numerically, say, by evaluating $0.0001^{0.0001} = 0.999079$.

Local extreme values
by logarithmic differentiation.

$$y = x^x$$

$$\ln y = \ln x^x = x\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = 1\cdot x + x\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

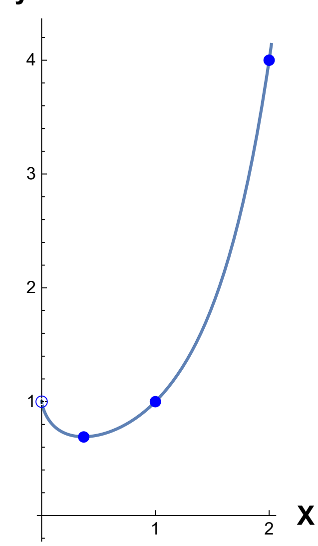
$$= x^x(\ln x + 1) = 0$$

$$\Rightarrow \ln x = -1$$

$$x = e^{-1} \doteq 0.37$$

$$y = 0.37^{0.37} \doteq 0.69$$

y = x^x



Another application of ***logarithmic differentiation*** is to the differentiation of some complicated terms.

Example

Find the derivative of $y = \frac{(x^2+1)(x+2)^3}{(x+4)^5}$.

$\ln y = \ln((x^2+1) + 3\ln(x+2) - 5\ln(x+4)$ — Properties of logs

$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{3}{x+2} - \frac{5}{x+4}$ — Logarithmic Differentiation

$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{3}{x+2} - \frac{5}{x+4}\right)$

$= \frac{(x^2+1)(x+2)^3}{(x+4)^5}\left(\frac{2x}{x^2+1} + \frac{3}{x+2} - \frac{5}{x+4}\right)$

This method is quick and produces a good looking answer.

Exercises 6.3

1. $\frac{d}{dx}(e^{2x+3}) =$ $\quad \frac{d}{dx}(e^{x^2}) =$ $\quad \frac{d}{dx}(e^{(2x+3)^5}) =$ $\quad \frac{d}{dx}(e^{\sin x}) =$

   $\frac{d}{dx}(e^x \tan x) =$ $\quad \frac{d}{dx}\left(\frac{e^{2x}}{x^2-3}\right) =$ $\quad \frac{d}{dx}(\cos(e^{2x+3})) =$ $\quad \frac{d}{dx}(x^3 e^{2x+3}) =$

2. $\int \frac{x}{x^2+1}\,dx =$ $\quad \int_0^1 \frac{x}{x^2+1}\,dx =$ $\quad \int \frac{\cos x}{\sin^2 x+1}\,dx =$ $\quad \int \frac{\cos x}{\sin}\,dx =$

3. $\int e^t\,dt =$ $\quad \int_0^{\pi} \cos x\, e^{\sin x}\,dx =$ $\quad \int 7^x\,dx =$ $\quad \int x\,10^{x^2}\,dx =$

4. Prove Property 2 of Logarithms $\mathbf{\ln \frac{x}{y} = \ln x - \ln y}$.

5. Prove Property 3 of Logarithms $\mathbf{\ln x^y = y \ln x}$.

6. $\frac{d}{dx}(\ln \frac{1}{x}) =$ $\quad \frac{d}{dx}(\ln x^2) =$ $\quad \frac{d}{dx}(\ln x^3) =$ $\quad \frac{d}{dx}(\ln x^n) =$

7. $\frac{d}{dx}(2^{3x}) =$ $\quad \frac{d}{dx}(10^{x^2}) =$ $\quad \frac{d}{dx}(7^{(2x+3)^5}) =$ $\quad \frac{d}{dx}(2^{\sin x}) =$

8. Work by logarithmic differentiation.

   $\frac{d}{dx}(x^{3x}) =$ $\quad \frac{d}{dx}(\sin x^{\cos x}) =$

9. Work by logarithmic differentiation. Compare with ordinary differentiation.

   a. $y = x(x+3)(x+5)$ $\quad$ b. $y = \frac{x(x^2+3)(x^3-5)}{x^4+2x^2+1}$

10. Write $y = 10^x$ in base e form.

Solutions

8 b. $y = \sin x^{\cos x}$

$\ln y = \cos x \ln(\sin x)$

$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln(\sin x) + \cos x \frac{\cos x}{\sin x}$

$\frac{dy}{dx} = y\left(\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x)\right)$

$= \sin x^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x)\right)$

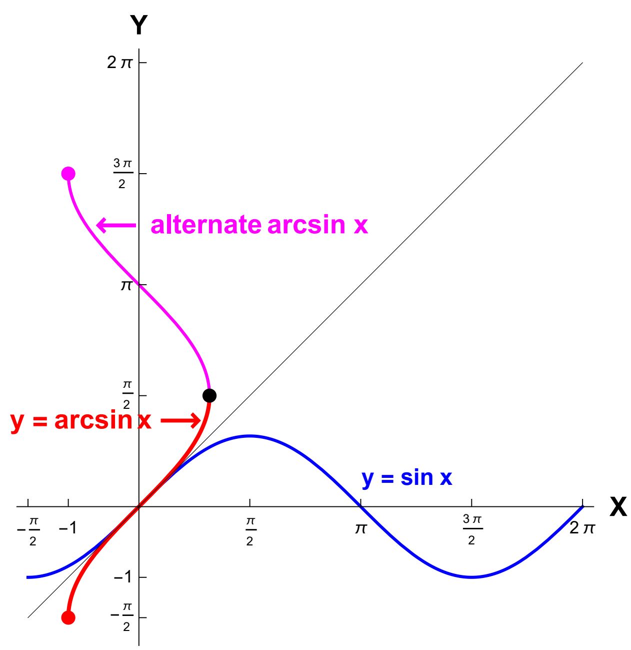
10. $10^x = \left(e^{\ln 10}\right)^x = e^{x \ln 10}$

6.4 Inverse Trig Functions: Algebra

The Inverse Sine Function

If you want to solve $\sin\theta = 0.347$, you need the inverse sine function: $\theta = \sin^{-1} 0.347 \doteq 20.304°$. The inverse sine function has many other applications in advanced calculus based applications.

Note: from the graph below, the function $y = \sin x$ is not one-to-one. The best we can do is observe that it is one-to-one, for example, on the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. We call the inverse function for this interval $\sin^{-1} x$ (if you are a mathematician) or for everyone else $\arcsin x$ (who might forget that $\sin^{-1} x$ does not mean $1/\sin x$.) If you chose the magenta colored function your answers would be between $90°$ and $270°$ (OK, but nobody does that!)

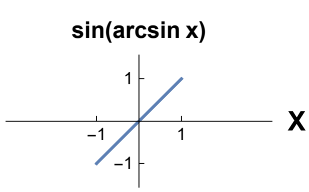


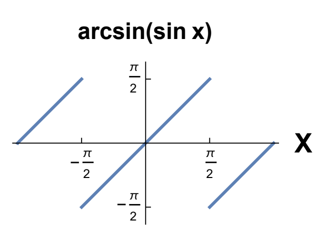
Inverse Trig Identities

$\sin(\arcsin x) = x, \ -1 \le x \le 1$ $\qquad$ $\arcsin(\sin x) = x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Examples ***Be careful*** : see below

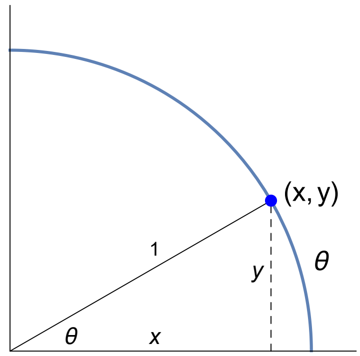
$\sin(\arcsin 2)$ DNE $\qquad$ $\arcsin(\sin \pi) = 0$





These examples are correct. But some math users always take $\sin(\arcsin x) = x$ and $\arcsin(\sin x) = x$ regardless of domains, but that could cause some serious mistakes.

Origin of the arcsin notation Look at the unit circle below.



$y = \sin\theta$

means

θ is an angle whose sine is y

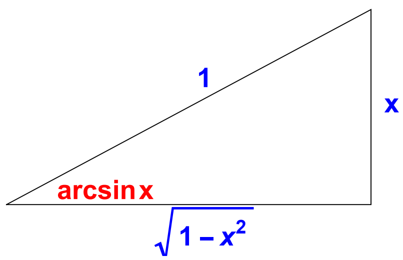
which means

θ is an arc whose sine is y

which said quickly is

$\theta = \arcsin y$

Other Inverse Trig Identities $\arcsin x$ means an 'angle whose sine is x'. It is often useful to construct a triangle illustrating this.



Picture of arcsin x

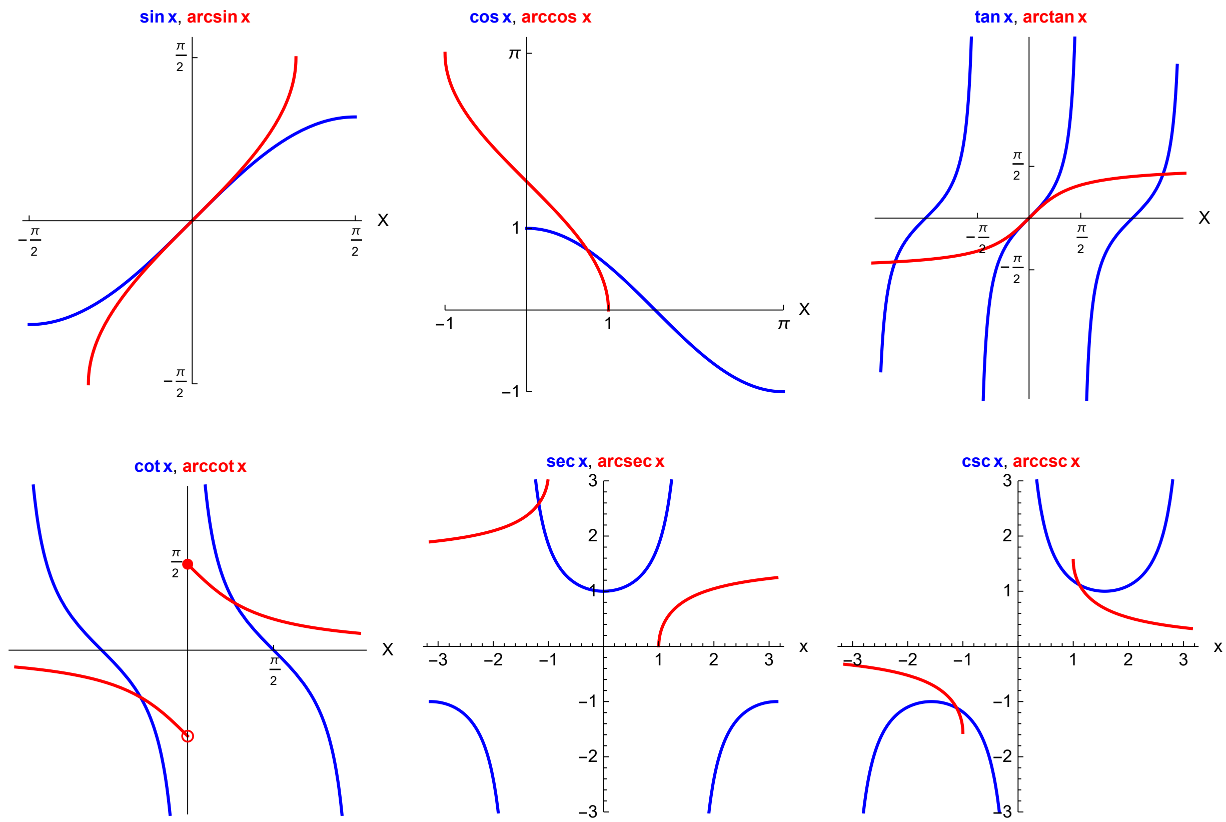
Full basic list of inverse sine identities (domains not shown) See picture above.

$$\sin(\arcsin x) = x \qquad \cos(\arcsin x) = \sqrt{1-x^2}$$

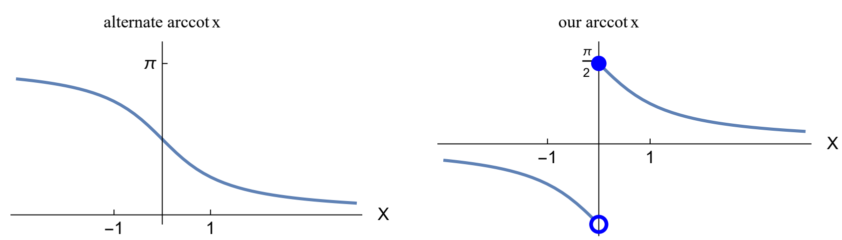
$$\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}} \qquad \cot(\arcsin x) = \frac{\sqrt{1-x^2}}{x}$$

$$\sec(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \csc(\arcsin x) = \frac{1}{x}$$

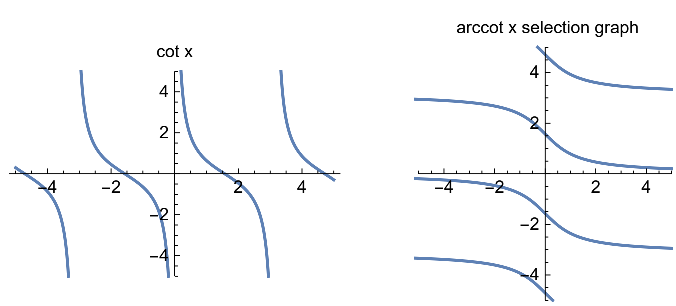
Basic Trig/ Inverse Trig Graphs, standard choices



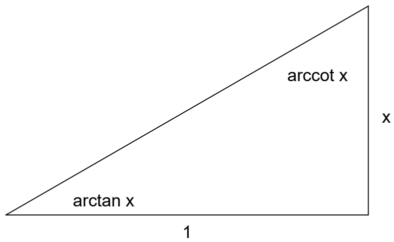
Alternate Choices. Warning Some textbooks and computer algebra systems may also choose arccot, arcsec and arccsc differently! (You will see in evaluating integrals, an important use of inverse trig functions, any reasonable choice leads to the same answer.)

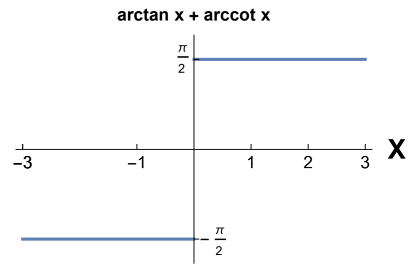


Note how both choices can be made.



One would expect: ArcTan[x] + ArcCot[x] $=\frac{\pi}{2}$. Why? See the diagram below.





Is there a geometry problem when $x < 0$?

Exercises 6.4

1. Solve using inverse trig functions. Evaluate by calculator.

   a. $\sin\theta = .35$ b. $\cos x = \frac{\pi}{13}$ c. $\tan y = 55$ d. $\sec x = 2$

2. Find the first three positive solutions of $\sin\theta = 0.3$ using inverse trig functions. Hint: Graph $\sin\theta$.

3. As in the first example, evaluate and check with a graph.

   a. cos(arccos 2) b. arccos(cos π)

4. Graph y = tan(arctan x) and y = arctan(tan x). Indicate domain and range of each.

5. State the **full basic list of inverse cosine identities**. Illustrate. State domains.

6. State the **full basic list of inverse tangent identities**. Illustrate. State domains.

7. Solve each.

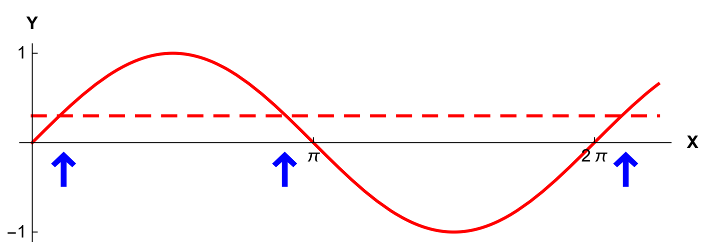
8. a. $\sin\theta = .358$. b. $\cos x = \frac{\pi}{13}$ c. $\tan y = 55$ g. d. $\sec x = 2$

9. a. sin(arcsin 2x) b. tan(arcsin 3x) c. sin(2arcsin x)

10. Graph and compare both sides of **cos(arcsin x)** $= \sqrt{1 - x^2}$ **.**

Solutions

2.



arcsin(0.3), π - arcsin(0.3), 2π + arcsin(0.3)

3. a. D b. π

9. (with domain restrictions)

a. sin(arcsin(2x))

$= 2x$

b. tan (arcsin(3x)

$= \frac{3x}{\sqrt{1-9x^2}}$

c. sin(2arcsin x)

$= 2\sin(\arcsin x)\cos(\arcsin x)$ double angle formula

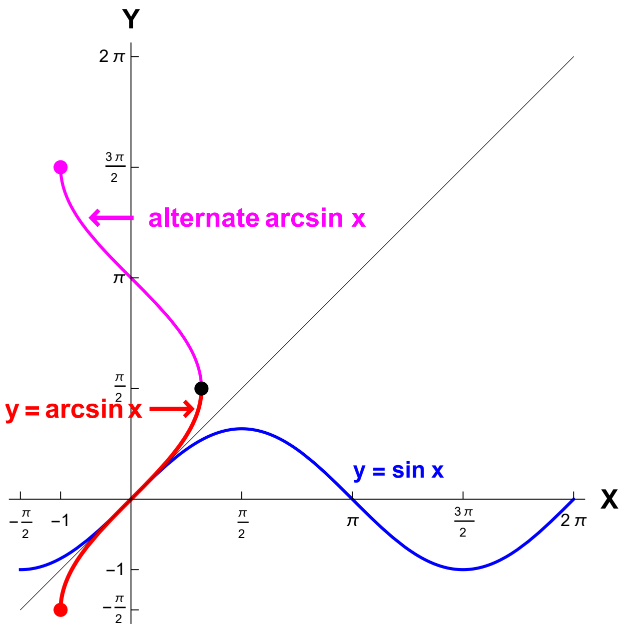
$= 2x\frac{\sqrt{1-x^2}}{1}$

$= 2x\sqrt{1-x^2}$

6.5 Calculus of the Inverse Trig Functions

Let us do $\frac{d}{dx}(\arcsin x)$ first.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$



Proof

$$y = \arcsin x \iff x = \sin y$$

$$1 = \cos y \frac{dy}{dx} \qquad \text{differentiating implicitly}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\pm\sqrt{1-\sin^2 y}} \qquad \text{Pythagorean Identity}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} \qquad \text{Since our } \arcsin x \text{ has a positive slope}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

For the alternate arcsin x we would have chosen the negative square root.

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Proof

$$y = \arctan x \iff x = \tan y$$

$$1 = \sec^2 y \frac{dy}{dx} \qquad \text{differentiating implicitly}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+\tan^2 y} \qquad \text{Pythagorean Identity}$$

$$= \frac{1}{1+x^2}$$

$\frac{d}{dx}(\text{arcsec } x) = \frac{1}{|x|\sqrt{x^2-1}}$

Proof

$y = \text{arcsec } x \iff x = \sec y$

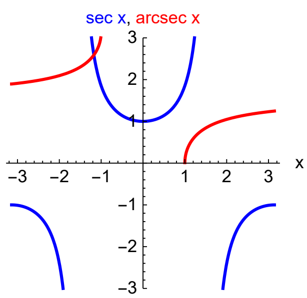
$1 = \sec y \tan y \frac{dy}{dx}$ differentiating implicitly

$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$

$= \frac{1}{\pm \sec y \sqrt{\tan^2 y - 1}}$ Pythagorean Identity

$= \frac{1}{\pm x \sqrt{x^2 - 1}}$

$= \frac{1}{|x| \sqrt{x^2 - 1}}$ See graph below



The slope of this arcsec x is always positive.

$\frac{d}{dx}(\text{arcsin } x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\text{arccos } x) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\text{arctan } x) = \frac{1}{1+x^2}$ $\frac{d}{dx}(\text{arccot } x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\text{arcsec } x) = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}(\text{arccsc } x) = \frac{-1}{|x|\sqrt{x^2-1}}$

$$\int \frac{dx}{\sqrt{1-x^2}} = \text{arcsin } x + C$$

$$\int \frac{dx}{1+x^2} = \text{arctan } x + C$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \text{arcsec } |x| + C$$

Why don't we turn the other three inverse trig derivatives into integral formulas?

Exercises 6.5

1. Do a full derivation of the derivative of arccos x. Show a relevant graph.

2. $\frac{d}{dx}(\arcsin x^2) =$ $\quad$ $\frac{d}{dx}(\sin x \arcsin x) =$ $\quad$ $\frac{d}{dx}\left(x \arcsin(ax + b)\right) =$

3. $\int \frac{dt}{\sqrt{1-t^2}} =$ $\quad$ $\int \frac{dx}{4+x^2} =$ $\quad$ $\int \frac{\cos x \, dx}{4+\sin^2 x} =$

4. The inverse trigonometric integrals are often written

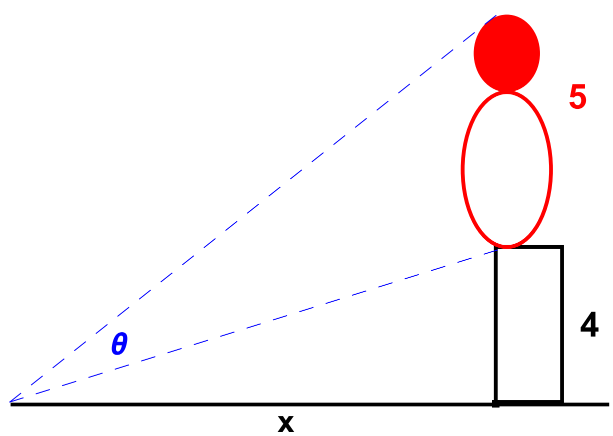
$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C.$$

Derive these.

5. A statue 5 meters high sits on a pedestal 4 meters high. Find the viewing angle θ as a function of the distance x. For what value of x is the viewing angle a maximum?



Solutions

5. Hint: $\theta = \arctan \frac{9}{x} - \arctan \frac{4}{x}$

Maximize θ:

Differentiate

Simplify

Solve

x = 6 meters.

6.6 Hyperbolic Functions

The **hyperbolic functions** are a set of functions that have many applications to mathematics, physics, and engineering. Among many other applications, they are used to describe the formation of satellite rings around planets, to describe the shape of a rope hanging from two points, and have application to the theory of special relativity. This section defines the hyperbolic functions and describes many of their properties, especially their usefulness to calculus.

These functions are sometimes referred to as the "hyperbolic trigonometric functions" as there are many, many connections between them and the standard trigonometric functions. Figure 6.6.1 demonstrates one such connection. Just as cosine and sine are used to define points on the circle defined by $x^2+y^2=1$, the functions **hyperbolic cosine** and **hyperbolic sine** are used to define points on the hyperbola $x^2-y^2=1$.

We begin with their definitions.

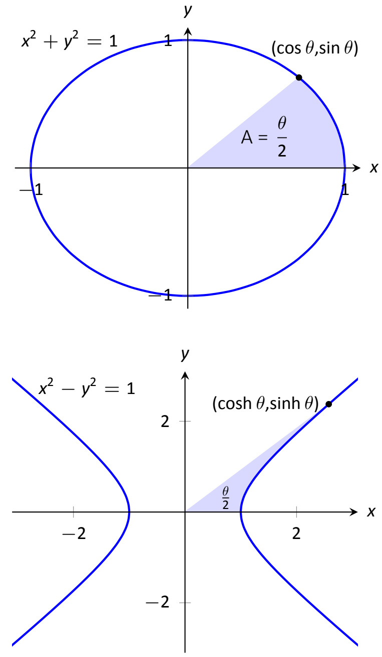


Figure 6.6.1: Using trigonometric functions to define points on a circle and hyperbolic functions to define points on a hyperbola. The area of the shaded regions are included in them.

Definition 6.6.1 Hyperbolic Functions

1. $\cosh x = \dfrac{e^x+e^{-x}}{2}$
2. $\sinh x = \dfrac{e^x-e^{-x}}{2}$
3. $\tanh x = \dfrac{\sinh x}{\cosh x}$
4. $\operatorname{sech} x = \dfrac{1}{\cosh x}$
5. $\operatorname{csch} x = \dfrac{1}{\sinh x}$
6. $\coth x = \dfrac{\cosh x}{\sinh x}$

These hyperbolic functions are graphed in Figure 6.6.2. In the graphs of $\cosh x$ and $\sinh x$, graphs of $e^x/2$ and $e^{-x}/2$ are included with dashed lines. As x gets "large," $\cosh x$ and $\sinh x$ each act like $e^x/2$; when x is a large negative number, $\cosh x$ acts like $e^{-x}/2$ whereas $\sinh x$ acts like $-e^{-x}/2$.

Notice the domains of $\tanh x$ and $\operatorname{sech} x$ are $(-\infty, \infty)$, whereas both $\coth x$ and $\operatorname{csch} x$ have vertical asymptotes at $x=0$. Also note the ranges of these functions, especially $\tanh x$: as $x\to\infty$, both $\sinh x$ and $\cosh x$ approach $e^x/2$, hence $\tanh x$ approaches 1.

The following example explores some of the properties of these functions that bear remarkable resemblance to the properties of their trigonometric counterparts.

Pronunciation Note:
"cosh" rhymes with "gosh,"
"sinh" rhymes with "pinch," and
"tanh" rhymes with "ranch."
"coth" rhymes with "crotch."

The connection between the trigonometric and hyperbolic functions shown in Figure 6.6.1 is correct but not particularly enlightening. We would like to see the relationships between the 'angles' rather than the areas.

In more advanced calculus that includes the imaginary number $i=\sqrt{-1}$, the relationships becomes clearer. That calculus is called *Functions of a Complex Variable*.

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\uparrow$$

$$e^x$$

$$\downarrow$$

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

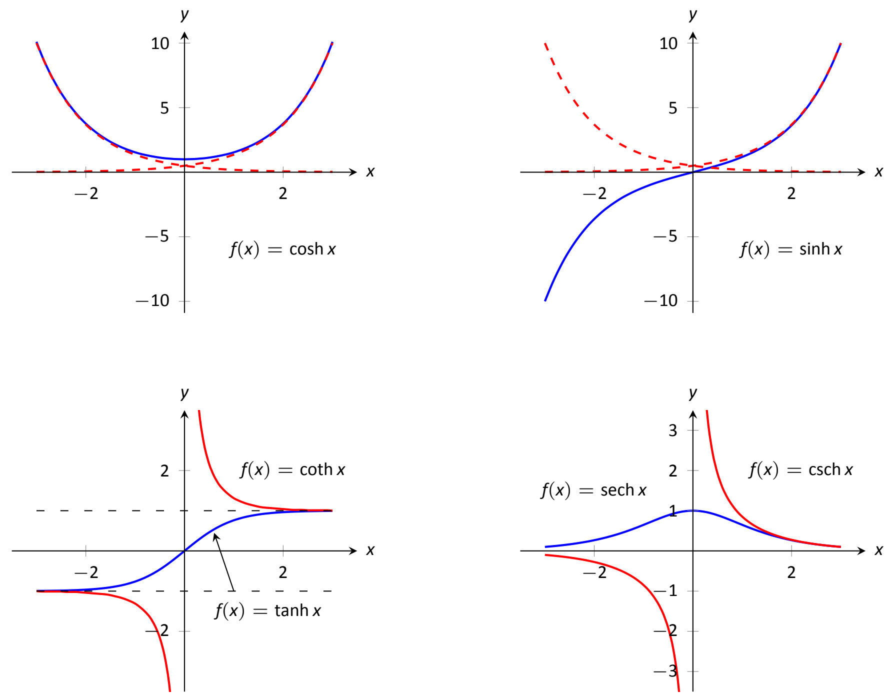


Figure 6.6.2: Graphs of the hyperbolic functions.

Example 6.6.1 Exploring properties of hyperbolic functions

Use Definition 6.6.1 to rewrite the following expressions.

1. $\cosh^2 x - \sinh^2 x$
2. $\tanh^2 x + \operatorname{sech}^2 x$
3. $2\cosh x \sinh x$
4. $\frac{d}{dx}\big(\cosh x\big)$
5. $\frac{d}{dx}\big(\sinh x\big)$
6. $\frac{d}{dx}\big(\tanh x\big)$

Solution

1.
$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}\\ &= \frac{4}{4} = 1.\end{aligned}$$

So $\cosh^2 x - \sinh^2 x = 1$.

2.
$$\begin{aligned}\tanh^2 x + \operatorname{sech}^2 x &= \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x}\\ &= \frac{\sinh^2 x + 1}{\cosh^2 x} \qquad \text{Now use identity from \#1.}\\ &= \frac{\cosh^2 x}{\cosh^2 x} = 1.\end{aligned}$$

So $\tanh^2 x + \operatorname{sech}^2 x = 1$.

3.
$$\begin{aligned}2\cosh x \sinh x &= 2\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)\\ &= 2 \cdot \frac{e^{2x} - e^{-2x}}{4}\\ &= \frac{e^{2x} - e^{-2x}}{2} = \sinh(2x).\end{aligned}$$

Thus $2\cosh x \sinh x = \sinh(2x)$.

4.
$$\begin{aligned}\frac{d}{dx}\big(\cosh x\big) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)\\ &= \frac{e^x - e^{-x}}{2}\\ &= \sinh x.\end{aligned}$$

So $\frac{d}{dx}\big(\cosh x\big) = \sinh x$.

5.
$$\begin{aligned}\frac{d}{dx}\big(\sinh x\big) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)\\ &= \frac{e^x + e^{-x}}{2}\\ &= \cosh x.\end{aligned}$$

So $\frac{d}{dx}\big(\sinh x\big) = \cosh x$.

6.
$$\begin{aligned}\frac{d}{dx}\big(\tanh x\big) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)\\ &= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}\\ &= \operatorname{sech}^2 x.\end{aligned}$$

So $\frac{d}{dx}\big(\tanh x\big) = \operatorname{sech}^2 x$.

The following Key Idea summarizes many of the important identities relating to hyperbolic functions. Each can be verified by referring back to Definition 6.6.1.

Key Idea 6.6.1 Useful Hyperbolic Function Properties. **You actually should semi-memorize these!**

Basic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\tanh^2 x + \text{sech}^2 x = 1$
3. $\coth^2 x - \text{csch}^2 x = 1$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x$
5. $\sinh 2x = 2\sinh x \cosh x$
6. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
7. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Derivatives

1. $\frac{d}{dx}\big(\cosh x\big) = \sinh x$
2. $\frac{d}{dx}\big(\sinh x\big) = \cosh x$
3. $\frac{d}{dx}\big(\tanh x\big) = \text{sech}^2 x$
4. $\frac{d}{dx}\big(\text{sech}\, x\big) = -\,\text{sech}\, x \tanh x$
5. $\frac{d}{dx}\big(\text{csch}\, x\big) = -\,\text{csch}\, x \coth x$
6. $\frac{d}{dx}\big(\coth x\big) = -\,\text{csch}^2 x$

Integrals

1. $\displaystyle\int \cosh x\, dx = \sinh x + C$
2. $\displaystyle\int \sinh x\, dx = \cosh x + C$
3. $\displaystyle\int \tanh x\, dx = \ln(\cosh x) + C$
4. $\displaystyle\int \coth x\, dx = \ln|\sinh x| + C$

We practice using Key Idea 6.6.1.

Example 6.6.2 Derivatives and integrals of hyperbolic functions
Evaluate the following derivatives and integrals.

1. $\dfrac{d}{dx}\big(\cosh 2x\big)$
2. $\displaystyle\int \text{sech}^2(7t-3)\, dt$
3. $\displaystyle\int_0^{\ln 2} \cosh x\, dx$

Solution

1. Using the Chain Rule directly, we have $\frac{d}{dx}\big(\cosh 2x\big) = 2\sinh 2x$.
   Just to demonstrate that it works, let's also use the Basic Identity found in Key Idea 6.6.1: $\cosh 2x = \cosh^2 x + \sinh^2 x$.

$$\begin{aligned}\frac{d}{dx}\big(\cosh 2x\big) = \frac{d}{dx}\big(\cosh^2 x + \sinh^2 x\big) &= 2\cosh x \sinh x + 2\sinh x \cosh x\\ &= 4\cosh x \sinh x.\end{aligned}$$

Using another Basic Identity, we can see that $4\cosh x \sinh x = 2\sinh 2x$. We get the same answer either way.

2. We employ substitution, with $u = 7t - 3$ and $du = 7dt$. Applying Key Ideas 6.1.1 and 6.6.1 we have:

$$\int \operatorname{sech}^2(7t-3)\,dt = \frac{1}{7}\tanh(7t-3) + C.$$

3. $$\int_0^{\ln 2} \cosh x\,dx = \sinh x\Big|_0^{\ln 2} = \sinh(\ln 2) - \sinh 0 = \sinh(\ln 2).$$

We can simplify this last expression as $\sinh x$ is based on exponentials:

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - 1/2}{2} = \frac{3}{4}.$$

Exercises 6.6

Terms and Concepts

1. In Key Idea 6.6.1, the equation $\int \tanh x \, dx = \ln(\cosh x) + C$ is given. Why is "$\ln|\cosh x|$" not used – i.e., why are absolute values not necessary?

2. The hyperbolic functions are used to define points on the right hand portion of the hyperbola $x^2 - y^2 = 1$, as shown in Figure 6.6.1. How can we use the hyperbolic functions to define points on the left hand portion of the hyperbola?

Problems

In Exercises 3 – 10, verify the given identity using Definition 6.6.1, as done in Example 6.6.1.

3. $\coth^2 x - \operatorname{csch}^2 x = 1$

4. $\cosh 2x = \cosh^2 x + \sinh^2 x$

5. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

6. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

7. $\dfrac{d}{dx}\left[\operatorname{sech} x\right] = -\operatorname{sech} x \tanh x$

8. $\dfrac{d}{dx}\left[\coth x\right] = -\operatorname{csch}^2 x$

9. $\displaystyle\int \tanh x \, dx = \ln(\cosh x) + C$

10. $\displaystyle\int \coth x \, dx = \ln|\sinh x| + C$

In Exercises 11 – 22, find the derivative of the given function.

11. $f(x) = \sinh 2x$

12. $f(x) = \cosh^2 x$

13. $f(x) = \tanh(x^2)$

14. $f(x) = \ln(\sinh x)$

15. $f(x) = \sinh x \cosh x$

16. $f(x) = x \sinh x - \cosh x$

In Exercises 23 – 28, find the equation of the line tangent to the function at the given *x*-value.

23. $f(x) = \sinh x$ at $x = 0$

24. $f(x) = \cosh x$ at $x = \ln 2$

25. $f(x) = \tanh x$ at $x = -\ln 3$

26. $f(x) = \operatorname{sech}^2 x$ at $x = \ln 3$

In Exercises 29 – 44, evaluate the given indefinite integral.

29. $\displaystyle\int \tanh(2x) \, dx$

30. $\displaystyle\int \cosh(3x - 7) \, dx$

31. $\displaystyle\int \sinh x \cosh x \, dx$

In Exercises 45 – 48, evaluate the given definite integral.

45. $\displaystyle\int_{-1}^{1} \sinh x \, dx$

46. $\displaystyle\int_{-\ln 2}^{\ln 2} \cosh x \, dx$

47. sech rhymes with?
csch rhymes with?

Solutions 6.6

1. Because $\cosh x$ is always positive.

2. The points on the left hand side can be defined as $(-\cosh x, \sinh x)$.

3. $$\begin{aligned}\coth^2 x - \operatorname{csch}^2 x &= \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2 - \left(\frac{2}{e^x-e^{-x}}\right)^2\\ &= \frac{(e^{2x}+2+e^{-2x})-(4)}{e^{2x}-2+e^{-2x}}\\ &= \frac{e^{2x}-2+e^{-2x}}{e^{2x}-2+e^{-2x}}\\ &= 1\end{aligned}$$

4. $$\begin{aligned}\cosh^2 x + \sinh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2 + \left(\frac{e^x-e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}+2+e^{-2x}}{4} + \frac{e^{2x}-2+e^{-2x}}{4}\\ &= \frac{2e^{2x}+2e^{-2x}}{4}\\ &= \frac{e^{2x}+e^{-2x}}{2}\\ &= \cosh 2x.\end{aligned}$$

5. $$\begin{aligned}\cosh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}+2+e^{-2x}}{4}\\ &= \frac{1}{2}\,\frac{(e^{2x}+e^{-2x})+2}{2}\\ &= \frac{1}{2}\left(\frac{e^{2x}+e^{-2x}}{2}+1\right)\\ &= \frac{\cosh 2x+1}{2}.\end{aligned}$$

6. $$\begin{aligned}\sinh^2 x &= \left(\frac{e^x-e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}-2+e^{-2x}}{4}\\ &= \frac{1}{2}\,\frac{(e^{2x}+e^{-2x})-2}{2}\\ &= \frac{1}{2}\left(\frac{e^{2x}+e^{-2x}}{2}-1\right)\\ &= \frac{\cosh 2x-1}{2}.\end{aligned}$$

7. $$\begin{aligned}\frac{d}{dx}\left[\operatorname{sech} x\right] &= \frac{d}{dx}\left[\frac{2}{e^x+e^{-x}}\right]\\ &= \frac{-2(e^x-e^{-x})}{(e^x+e^{-x})^2}\\ &= -\frac{2(e^x-e^{-x})}{(e^x+e^{-x})(e^x+e^{-x})}\\ &= -\frac{2}{e^x+e^{-x}}\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}\\ &= -\operatorname{sech} x\tanh x\end{aligned}$$

8. $$\begin{aligned}\frac{d}{dx}\left[\coth x\right] &= \frac{d}{dx}\left[\frac{e^x+e^{-x}}{e^x-e^{-x}}\right]\\ &= \frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}\\ &= \frac{e^{2x}+e^{-2x}-2-(e^{2x}+e^{-2x}+2)}{(e^x-e^{-x})^2}\\ &= -\frac{4}{(e^x-e^{-x})^2}\\ &= -\operatorname{csch}^2 x\end{aligned}$$

9. $$\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx$$
Let $u=\cosh x$; $du=(\sinh x)dx$
$$\begin{aligned} &= \int \frac{1}{u}\,du\\ &= \ln|u|+C\\ &= \ln(\cosh x)+C.\end{aligned}$$

10. $$\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx$$
Let $u=\sinh x$; $du=(\cosh x)dx$
$$\begin{aligned} &= \int \frac{1}{u}\,du\\ &= \ln|u|+C\\ &= \ln|\sinh x|+C.\end{aligned}$$

11. $2\cosh 2x$

12. Taking the derivative of $(\cosh x)^2$ directly, one gets $2\cosh x\sinh x$; using the identity $\cosh^2 x = \frac{1}{2}(\cosh 2x+1)$ first, one gets $\sinh 2x$; by Key Idea 6.6.1, these are equal.

13. $2x\sec^2(x^2)$

14. $\coth x$

15. $\sinh^2 x + \cosh^2 x$

16. $x\cosh x$

23. $y = x$

24. $y = (x-\ln 2)+\frac{5}{4}$

25. $y = \frac{9}{25}(x+\ln 3)-\frac{4}{5}$

26. $y = -\frac{72}{125}(x-\ln 3)+\frac{9}{25}$

29. $1/2\ln(\cosh(2x))+C$

30. $1/3\sinh(3x-7)+C$

31. $1/2\sinh^2 x + C$ or $1/2\cosh^2 x + C$

45. 0

46. $3/2$

47. screetch
go screetch

6.7 Inverse Hyperbolic Functions

Just as the inverse trigonometric functions are useful in certain integrations, the inverse hyperbolic functions are useful with other related ones. Figure 6.6.3 shows the restrictions on the domains to make each function one-to-one and the resulting domains and ranges of their inverse functions. Their graphs are shown in Figure 6.6.4.

Because the hyperbolic functions are defined in terms of exponential functions, their inverses can be expressed in terms of logarithms as shown in Key Idea 6.6.2. It is often more convenient to refer to $\sinh^{-1} x$ than to $\ln\left(x+\sqrt{x^2+1}\right)$, especially when one is working on theory and does not need to compute actual values. On the other hand, when computations are needed, technology is often helpful but many hand-held calculators lack a *convenient* $\sinh^{-1} x$ button. (Often it can be accessed under a menu system, but not conveniently.) In such a situation, the logarithmic representation is useful. ***The reader is not encouraged to memorize these, but rather know they exist and know how to use them when needed, i.e., semi-memorize them.***

Function	Domain	Range
$\cosh x$	$[0,\infty)$	$[1,\infty)$
$\sinh x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\tanh x$	$(-\infty,\infty)$	$(-1,1)$
~~$\operatorname{sech} x$~~	~~$[0,\infty)$~~	~~$(0,1]$~~
$\operatorname{csch} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\coth x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,-1)\cup(1,\infty)$

Function	Domain	Range
$\cosh^{-1} x$	$[1,\infty)$	$[0,\infty)$
$\sinh^{-1} x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\tanh^{-1} x$	$(-1,1)$	$(-\infty,\infty)$
$\operatorname{sech}^{-1} x$	$(0,1]$	$[0,\infty)$
$\operatorname{csch}^{-1} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\coth^{-1} x$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$

Figure 6.7.3: Domains and ranges of the hyperbolic and inverse hyperbolic functions.

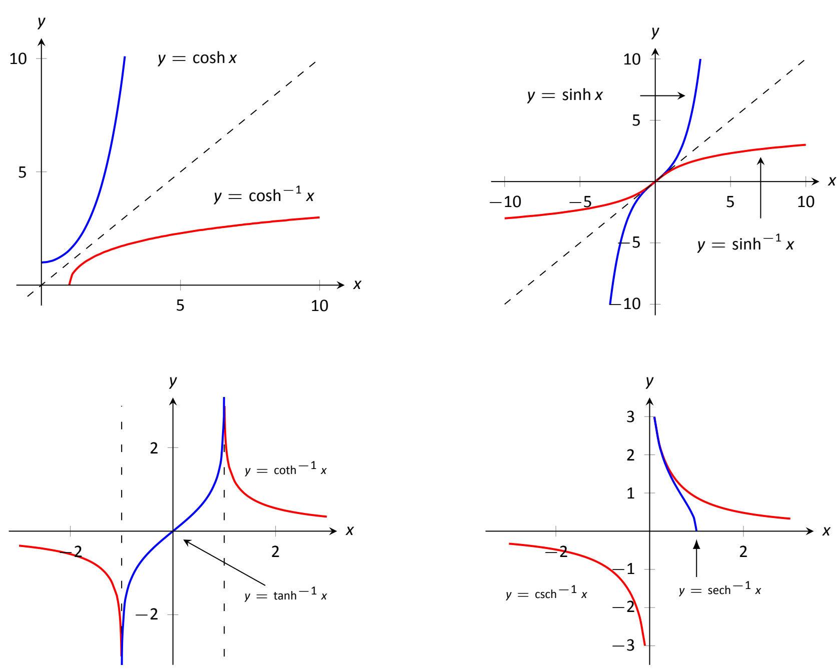


Figure 6.6.4: Graphs of the hyperbolic functions and their inverses.

Note on Notation

The '-1 power' notation is widely used by mathematicians. This seems to make sense:

y = f(x) ⇔ x = $f^{-1}(y)$.

It looks like to go from the left equation to the right one is take $f^{-1} \stackrel{?}{=} \frac{1}{f}$. But reciprocals are for variables not functions. Many beginners make the mistake of writing $\sinh^{-1}(x) = \frac{1}{\sinh x}$. Bad. Bad. Otherwise the notation $\sinh^{-1}$ is just fine.

Some authors write arcsinh, but arcsinh has nothing much to do with arcs or angles.

Others write argsinh because in y = sinh x, x is the ***argument*** of the sinh function. It is a good notation but not widely used.

By Calculus II you should be able to handle $\sinh^{-1}$!

Key Idea 6.6.2 Logarithmic definitions of Inverse Hyperbolic Functions

1. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right);\ x \geq 1$
2. $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right);\ |x| < 1$
3. $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right);\ 0 < x \leq 1$
4. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$
5. $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right);\ |x| > 1$
6. $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right);\ x \neq 0$

The following Key Ideas give the derivatives and integrals relating to the inverse hyperbolic functions. In Key Idea 6.6.4, both the inverse hyperbolic and logarithmic function representations of the antiderivative are given, based on Key Idea 6.6.2. Again, these latter functions are often more useful than the former. Note how inverse hyperbolic functions can be used to solve integrals we will use Trigonometric Substitution to solve in Section 7.4.

Key Idea 6.6.3 Derivatives Involving Inverse Hyperbolic Functions

1. $\dfrac{d}{dx}\left(\cosh^{-1}x\right)=\dfrac{1}{\sqrt{x^2-1}};\ x>1$

2. $\dfrac{d}{dx}\left(\sinh^{-1}x\right)=\dfrac{1}{\sqrt{x^2+1}}$

3. $\dfrac{d}{dx}\left(\tanh^{-1}x\right)=\dfrac{1}{1-x^2};\ |x|<1$

4. $\dfrac{d}{dx}\left(\operatorname{sech}^{-1}x\right)=\dfrac{-1}{x\sqrt{1-x^2}};0<x<1$

5. $\dfrac{d}{dx}\left(\operatorname{csch}^{-1}x\right)=\dfrac{-1}{|x|\sqrt{1+x^2}};\ x\neq 0$

6. $\dfrac{d}{dx}\left(\coth^{-1}x\right)=\dfrac{1}{1-x^2};\ |x|>1$

Key Idea 6.6.4 Integrals Involving Inverse Hyperbolic Functions

1. $\displaystyle\int\frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)+C;\ 0<a<x \quad = \ln\left|x+\sqrt{x^2-a^2}\right|+C$

2. $\displaystyle\int\frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)+C;\ a>0 \quad = \ln\left|x+\sqrt{x^2+a^2}\right|+C$

3. $\displaystyle\int\frac{1}{a^2-x^2}\,dx = \begin{cases}\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C & x^2<a^2\\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right)+C & a^2<x^2\end{cases} \quad = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$

4. $\displaystyle\int\frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{x}{a}\right)+C;\ 0<x<a \quad = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right)+C$

5. $\displaystyle\int\frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{x}{a}\right|+C;\ x\neq 0,\ a>0 \quad = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right|+C$

These we will use later.

We practice using a derivative formula in the following example.

Example 6.6.3 Derivative involving inverse hyperbolic functions

Evaluate.

$$\frac{d}{dx}\left[\cosh^{-1}\left(\frac{3x-2}{5}\right)\right]$$

SOLUTION

Applying Key Idea 6.6.3 with the Chain Rule gives:

$$\frac{d}{dx}\left[\cosh^{-1}\left(\frac{3x-2}{5}\right)\right]=\frac{1}{\sqrt{\left(\frac{3x-2}{5}\right)^2-1}}\cdot\frac{3}{5}.$$

We practice using an integral formula in the following example.

$$\int \frac{1}{x^2-1}\,dx$$

SOLUTION

Multiplying the numerator and denominator by (-1) gives: $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{-1}{1-x^2}\,dx$. The second integral can be solved with a direct application of item #3 from Key Idea 6.6.4, with $a = 1$. Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= -\int \frac{1}{1-x^2}\,dx \\ &= \begin{cases} -\tanh^{-1}(x) + C & x^2 < 1 \\ -\coth^{-1}(x) + C & 1 < x^2 \end{cases} \\ &= -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.\end{aligned}$$

This section covers a lot of ground. New functions were introduced, along with some of their fundamental identities, their derivatives and antiderivatives, their inverses, and the derivatives and antiderivatives of these inverses. Four Key Ideas were presented, each including quite a bit of information.

Do not view this section as containing a source of information to be memorized, but rather as a reference for future problem solving. Key Idea 6.6.4 contains perhaps the most useful information we will use later

Exercises 6.7

In Exercises 11 – 22, find the derivative of the given function.

17. $f(x) = \text{sech}^{-1}(x^2)$

18. $f(x) = \sinh^{-1}(3x)$

19. $f(x) = \cosh^{-1}(2x^2)$

20. $f(x) = \tanh^{-1}(x+5)$

21. $f(x) = \tanh^{-1}(\cos x)$

22. $f(x) = \cosh^{-1}(\sec x)$

In Exercises 23 – 28, find the equation of the line tangent to the function at the given x-value.

27. $f(x) = \sinh^{-1} x$ at $x = 0$

28. $f(x) = \cosh^{-1} x$ at $x = \sqrt{2}$

In Exercises 29 – 44, evaluate the given indefinite integral.

34. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$

35. $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$

36. $\displaystyle\int \frac{1}{9-x^2}\,dx$

Solutions 6.7

17. $\frac{-2x}{(x^2)\sqrt{1-x^4}}$

18. $\frac{3}{\sqrt{9x^2+1}}$

19. $\frac{4x}{\sqrt{4x^4-1}}$

20. $\frac{1}{1-(x+5)^2}$

21. $-\csc x$

22. $\sec x$

27. $y = x$

28. $y = (x-\sqrt{2}) + \cosh^{-1}(\sqrt{2}) \approx (x-1.414) + 0.881$

34. $\sinh^{-1} x + C = \ln\left(x + \sqrt{x^2+1}\right) + C$

35. $\cosh^{-1} x/3 + C = \ln\left(x + \sqrt{x^2-9}\right) + C$

36. $\begin{cases} \frac{1}{3}\tanh^{-1}\left(\frac{x}{3}\right) + C & x^2 < 9 \\ \frac{1}{3}\coth^{-1}\left(\frac{x}{3}\right) + C & 9 < x^2 \end{cases} =$
$\frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$

6.8 L'Hôpital's Rule

While this chapter is devoted to learning techniques of integration, this section is not about integration. Rather, it is concerned with a technique of evaluating certain limits that will be useful in the following section, where integration is once more discussed.

Our treatment of limits exposed us to the notion of "0/0", an indeterminate form. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, we do not conclude that $\lim_{x\to c} f(x)/g(x)$ is $0/0$; rather, we use $0/0$ as notation to describe the fact that both the numerator and denominator approach 0. The expression 0/0 has no numeric value; other work must be done to evaluate the limit.

Other indeterminate forms exist; they are: ∞/∞, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0. Just as "0/0" does not mean "divide 0 by 0," the expression "∞/∞" does not mean "divide infinity by infinity." Instead, it means "a quantity is growing without bound and is being divided by another quantity that is growing without bound." We cannot determine from such a statement what value, if any, results in the limit. Likewise, "$0\cdot\infty$" does not mean "multiply zero by infinity." Instead, it means "one quantity is shrinking to zero, and is being multiplied by a quantity that is growing without bound." We cannot determine from such a description what the result of such a limit will be.

This section introduces l'Hôpital's Rule, a method of resolving limits that produce the indeterminate forms 0/0 and ∞/∞. We'll also show how algebraic manipulation can be used to convert other indeterminate expressions into one of these two forms so that our new rule can be applied.

Theorem 6.8.1 L'Hôpital's Rule, Part 1

Let $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, where f and g are differentiable functions on an open interval I containing c, and $g'(x) \neq 0$ on I except possibly at c. Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Proof Near $x = c$, $x \neq c$ by tan line approximations

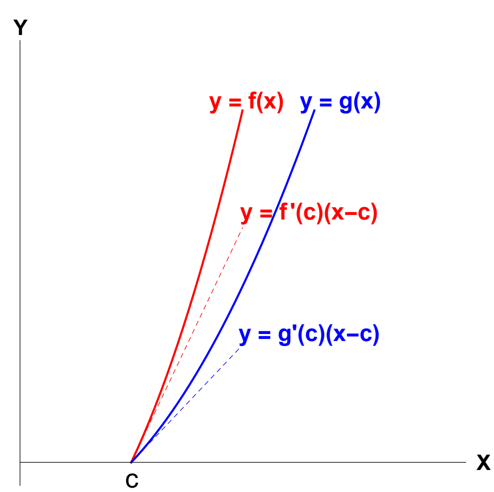
$$f(x) \approx f(c) + f'(c)(x - c) = f'(c)(x - c)$$
$$g(x) \approx g(c) + g'(c)(x - c) = g'(c)(x - c)$$

So

$$\frac{f(x)}{g(x)} \approx \frac{f'(c)(x-c)}{g'(c)(x-c)} \to \frac{f'(c)}{g'(c)}$$

or

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$



We demonstrate the use of l'Hôpital's Rule in the following examples; we will often use "LHR" as an abbreviation of "l'Hôpital's Rule."

Example 6.8.2 Using l'Hôpital's Rule

Evaluate the following limits, using l'Hôpital's Rule as needed.

1. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$
2. $\lim\limits_{x\to 1} \dfrac{\sqrt{x+3}-2}{1-x}$
3. $\lim\limits_{x\to 0} \dfrac{x^2}{1-\cos x}$
4. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x^2-3x+2}$

SOLUTION

1. We proved this limit is 1 in Example 1.3.4 using the Squeeze Theorem. Here we use l'Hôpital's Rule to show its power.

$$\lim_{x\to 0} \frac{\sin x}{x} \stackrel{\text{by LHR}}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

2.
$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x} \stackrel{\text{by LHR}}{=} \lim_{x\to 1} \frac{\frac{1}{2}(x+3)^{-1/2}}{-1} = -\frac{1}{4}.$$

3.
$$\lim_{x\to 0} \frac{x^2}{1-\cos x} \stackrel{\text{by LHR}}{=} \lim_{x\to 0} \frac{2x}{\sin x}.$$

This latter limit also evaluates to the 0/0 indeterminate form. To evaluate it, we apply l'Hôpital's Rule again.

$$\lim_{x\to 0} \frac{2x}{\sin x} \stackrel{\text{by LHR}}{=} \frac{2}{\cos x} = 2.$$

Thus $\lim\limits_{x\to 0} \dfrac{x^2}{1-\cos x} = 2.$

4. We already know how to evaluate this limit; first factor the numerator and denominator. We then have:

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+3}{x-1} = 5.$$

We now show how to solve this using l'Hôpital's Rule.

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2} \stackrel{\text{by LHR}}{=} \lim_{x\to 2} \frac{2x+1}{2x-3} = 5.$$

Note that at each step where l'Hôpital's Rule was applied, it was *needed*: the initial limit returned the indeterminate form of "$0/0$." If the initial limit returns, for example, 1/2, then l'Hôpital's Rule does not apply.

The following theorem extends our initial version of l'Hôpital's Rule in two ways. It allows the technique to be applied to the indeterminate form ∞/∞ and to limits where x approaches $\pm\infty$.

Theorem 6.8.2 L'Hôpital's Rule, Part 2

1. Let $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, where f and g are differentiable on an open interval I containing a. Then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$
2. Let f and g be differentiable functions on the open interval (a, ∞) for some value a, where $g'(x) \neq 0$ on (a, ∞) and $\lim_{x\to\infty} f(x)/g(x)$ returns either "0/0" or "∞/∞". Then
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$
A similar statement can be made for limits where x approaches $-\infty$.

Example 6.8.2 Using l'Hôpital's Rule with limits involving ∞
Evaluate the following limits.

1. $\lim_{x\to\infty} \dfrac{3x^2 - 100x + 2}{4x^2 + 5x - 1000}$
2. $\lim_{x\to\infty} \dfrac{e^x}{x^3}$.

SOLUTION

1. We can evaluate this limit already using Theorem 1.6.1; the answer is 3/4. We apply l'Hôpital's Rule to demonstrate its applicability.
$$\lim_{x\to\infty} \frac{3x^2 - 100x + 2}{4x^2 + 5x - 1000} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{6x - 100}{8x + 5} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{6}{8} = \frac{3}{4}.$$
2. $\lim_{x\to\infty} \dfrac{e^x}{x^3} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \dfrac{e^x}{3x^2} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \dfrac{e^x}{6x} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \dfrac{e^x}{6} = \infty.$
Recall that this means that the limit does not exist; as x approaches ∞, the expression e^x/x^3 grows without bound. We can infer from this that e^x grows "faster" than x^3; as x gets large, e^x is far larger than x^3. (This

has important implications in computing when considering efficiency of algorithms.)

Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

L'Hôpital's Rule can only be applied to ratios of functions. When faced with an indeterminate form such as $0 \cdot \infty$ or $\infty - \infty$, we can sometimes apply algebra to rewrite the limit so that l'Hôpital's Rule can be applied. We demonstrate the general idea in the next example.

Example 6.8.3 Applying l'Hôpital's Rule to other indeterminate forms
Evaluate the following limits.

1. $\displaystyle\lim_{x\to 0^+} x \cdot e^{1/x}$
2. $\displaystyle\lim_{x\to 0^-} x \cdot e^{1/x}$
3. $\displaystyle\lim_{x\to \infty} \ln(x+1) - \ln x$
4. $\displaystyle\lim_{x\to \infty} x^2 - e^x$

Solution

1. As $x \to 0^+$, $x \to 0$ and $e^{1/x} \to \infty$. Thus we have the indeterminate form $0 \cdot \infty$. We rewrite the expression $x \cdot e^{1/x}$ as $\dfrac{e^{1/x}}{1/x}$; now, as $x \to 0^+$, we get the indeterminate form ∞/∞ to which l'Hôpital's Rule can be applied.
$$\lim_{x\to 0^+} x \cdot e^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} \overset{\text{by LHR}}{=} \lim_{x\to 0^+} \frac{(-1/x^2)e^{1/x}}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty.$$
Interpretation: $e^{1/x}$ grows "faster" than x shrinks to zero, meaning their product grows without bound.
2. As $x \to 0^-$, $x \to 0$ and $e^{1/x} \to e^{-\infty} \to 0$. The the limit evaluates to $0 \cdot 0$ which is not an indeterminate form. We conclude then that
$$\lim_{x\to 0^-} x \cdot e^{1/x} = 0.$$
3. This limit initially evaluates to the indeterminate form $\infty - \infty$. By applying a logarithmic rule, we can rewrite the limit as
$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right).$$
As $x \to \infty$, the argument of the ln term approaches ∞/∞, to which we can apply l'Hôpital's Rule.
$$\lim_{x\to\infty} \frac{x+1}{x} \overset{\text{by LHR}}{=} \frac{1}{1} = 1.$$

Since $x \to \infty$ implies $\dfrac{x+1}{x} \to 1$, it follows that

$$x \to \infty \quad \text{implies} \quad \ln\left(\frac{x+1}{x}\right) \to \ln 1 = 0.$$

Thus

$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = 0.$$

Interpretation: since this limit evaluates to 0, it means that for large x, there is essentially no difference between $\ln(x+1)$ and $\ln x$; their difference is essentially 0.

4. The limit $\lim\limits_{x\to\infty} x^2 - e^x$ initially returns the indeterminate form $\infty - \infty$. We can rewrite the expression by factoring out x^2; $x^2 - e^x = x^2\left(1 - \dfrac{e^x}{x^2}\right)$. We need to evaluate how e^x/x^2 behaves as $x \to \infty$:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Thus $\lim_{x\to\infty} x^2(1 - e^x/x^2)$ evaluates to $\infty \cdot (-\infty)$, which is not an indeterminate form; rather, $\infty \cdot (-\infty)$ evaluates to $-\infty$. We conclude that $\lim\limits_{x\to\infty} x^2 - e^x = -\infty$.

Interpretation: as x gets large, the difference between x^2 and e^x grows very large.

Indeterminate Forms 0^0, 1^∞ and ∞^0

When faced with an indeterminate form that involves a power, it often helps to employ the natural logarithmic function. The following Key Idea expresses the concept, which is followed by an example that demonstrates its use.

Key Idea 6.8.1 Evaluating Limits Involving Indeterminate Forms 0^0, 1^∞ and ∞^0

If $\lim\limits_{x\to c} \ln\big(f(x)\big) = L$, then $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} e^{\ln(f(x))} = e^{L}$.

Example 6.8.4 Using l'Hôpital's Rule with indeterminate forms involving exponents

Evaluate the following limits.

1. $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$
2. $\displaystyle\lim_{x\to 0^+} x^x$.

SOLUTION

1. This is equivalent to a special limit given in Theorem 1.3.5; these limits have important applications within mathematics and finance. Note that the exponent approaches ∞ while the base approaches 1, leading to the indeterminate form 1^∞. Let $f(x) = (1+1/x)^x$; the problem asks to evaluate $\displaystyle\lim_{x\to\infty} f(x)$. Let's first evaluate $\displaystyle\lim_{x\to\infty}\ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to\infty}\ln\big(f(x)\big) &= \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x\\ &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)\\ &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\end{aligned}$$

This produces the indeterminate form 0/0, so we apply l'Hôpital's Rule.

$$\begin{aligned}&= \lim_{x\to\infty}\frac{\frac{1}{1+1/x}\cdot(-1/x^2)}{(-1/x^2)}\\ &= \lim_{x\to\infty}\frac{1}{1+1/x}\\ &= 1.\end{aligned}$$

Thus $\displaystyle\lim_{x\to\infty}\ln\big(f(x)\big) = 1$. We return to the original limit and apply Key Idea 6.8.1.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln(f(x))} = e^1 = e.$$

2. This limit leads to the indeterminate form 0^0. Let $f(x) = x^x$ and consider

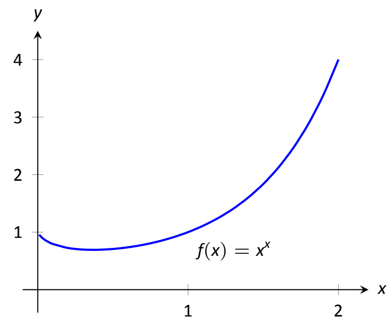


Figure 6.8.1: A graph of $f(x) = x^x$ suppporting the fact that as $x \to 0^+$, $f(x) \to 1$.

first $\lim\limits_{x\to 0^+} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to 0^+} \ln\big(f(x)\big) &= \lim_{x\to 0^+} \ln\left(x^x\right)\\ &= \lim_{x\to 0^+} x\ln x\\ &= \lim_{x\to 0^+} \frac{\ln x}{1/x}.\end{aligned}$$

This produces the indeterminate form $-\infty/\infty$ so we apply l'Hôpital's Rule.

$$\begin{aligned} &= \lim_{x\to 0^+} \frac{1/x}{-1/x^2}\\ &= \lim_{x\to 0^+} -x\\ &= 0.\end{aligned}$$

Thus $\lim\limits_{x\to 0^+} \ln\big(f(x)\big) = 0$. We return to the original limit and apply Key Idea 6.8.1.

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln(f(x))} = e^0 = 1.$$

This result is supported by the graph of $f(x) = x^x$ given in Figure 6.8.1.

Our brief revisit of limits will be rewarded in the next section where we consider *improper integration.* So far, we have only considered definite integrals where the bounds are finite numbers, such as $\int_0^1 f(x)\,dx$. Improper integration considers integrals where one, or both, of the bounds are "infinity." Such integrals have many uses and applications, in addition to generating ideas that are enlightening.

Alternate one step approach if you are fluent with logs for exponential forms.

$$\begin{aligned}
&\lim_{x\to 0^+} x^x && \{0^0\}\\
&= \lim_{x\to 0^+} e^{\ln x^x}\\
&= \lim_{x\to 0^+} e^{x\ln x}\ x && \{0\cdot\infty\}\\
&= \lim_{x\to 0^+} e^{\frac{\ln x}{1/x}} && \left\{\frac{\infty}{\infty}\right\}\\
&\overset{\text{LHR}}{=} \lim_{x\to 0^+} e^{\frac{1/x}{-1/x^2}}\\
&= \lim_{x\to 0^+} e^{-x}\\
&= e^0\\
&= 1
\end{aligned}$$

Exercises 6.8

Terms and Concepts

1. List the different indeterminate forms described in this section.

2. T/F: l'Hôpital's Rule provides a faster method of computing derivatives.

3. T/F: l'Hôpital's Rule states that $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)}{g'(x)}$.

4. Explain what the indeterminate form "1^∞" means.

5. Fill in the blanks:

   The Quotient Rule is applied to $\frac{f(x)}{g(x)}$ when taking _________;

   l'Hôpital's Rule is applied to $\frac{f(x)}{g(x)}$ when taking certain _________.

6. Create (but do not evaluate!) a limit that returns "∞^0".

7. Create a function $f(x)$ such that $\lim\limits_{x\to 1} f(x)$ returns "0^0".

8. Create a function $f(x)$ such that $\lim\limits_{x\to\infty} f(x)$ returns "$0 \cdot \infty$".

Problems 6.8

In Exercises 9 – 54, evaluate the given limit.

9. $\lim\limits_{x\to 1} \frac{x^2+x-2}{x-1}$

10. $\lim\limits_{x\to 2} \frac{x^2+x-6}{x^2-7x+10}$

11. $\lim\limits_{x\to \pi} \frac{\sin x}{x-\pi}$

12. $\lim\limits_{x\to \pi/4} \frac{\sin x - \cos x}{\cos(2x)}$

13. $\lim\limits_{x\to 0} \frac{\sin(5x)}{x}$

14. $\lim\limits_{x\to 0} \frac{\sin(2x)}{x+2}$

15. $\lim\limits_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$

16. $\lim\limits_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$

17. $\lim\limits_{x\to 0^+} \frac{e^x-1}{x^2}$

18. $\lim\limits_{x\to 0^+} \frac{e^x-x-1}{x^2}$

19. $\lim\limits_{x\to 0^+} \frac{x-\sin x}{x^3-x^2}$

20. $\lim\limits_{x\to \infty} \frac{x^4}{e^x}$

21. $\lim\limits_{x\to \infty} \frac{\sqrt{x}}{e^x}$

22. $\lim\limits_{x\to \infty} \frac{1}{x^2}e^x$

23. $\lim\limits_{x\to \infty} \frac{e^x}{\sqrt{x}}$

24. $\lim\limits_{x\to \infty} \frac{e^x}{2^x}$

25. $\lim\limits_{x\to \infty} \frac{e^x}{3^x}$

26. $\lim\limits_{x\to 3} \frac{x^3-5x^2+3x+9}{x^3-7x^2+15x-9}$

27. $\lim\limits_{x\to -2} \frac{x^3+4x^2+4x}{x^3+7x^2+16x+12}$

28. $\lim\limits_{x\to \infty} \frac{\ln x}{x}$

29. $\lim\limits_{x\to \infty} \frac{\ln(x^2)}{x}$

30. $\lim\limits_{x\to \infty} \frac{\left(\ln x\right)^2}{x}$

31. $\lim\limits_{x\to 0^+} x \cdot \ln x$

32. $\lim\limits_{x\to 0^+} \sqrt{x} \cdot \ln x$

33. $\lim\limits_{x\to 0^+} xe^{1/x}$

34. $\lim\limits_{x\to \infty} x^3 - x^2$

35. $\lim\limits_{x\to \infty} \sqrt{x} - \ln x$

36. $\lim\limits_{x\to -\infty} xe^x$

37. $\lim\limits_{x\to 0^+} \frac{1}{x^2}e^{-1/x}$

38. $\lim\limits_{x\to 0^+} (1+x)^{1/x}$

39. $\lim_{x\to 0^+} (2x)^x$
40. $\lim_{x\to 0^+} (2/x)^x$
41. $\lim_{x\to 0^+} (\sin x)^x$
42. $\lim_{x\to 1^+} (1-x)^{1-x}$
43. $\lim_{x\to \infty} (x)^{1/x}$
44. $\lim_{x\to \infty} (1/x)^x$
45. $\lim_{x\to 1^+} (\ln x)^{1-x}$
46. $\lim_{x\to \infty} (1+x)^{1/x}$
47. $\lim_{x\to \infty} (1+x^2)^{1/x}$
48. $\lim_{x\to \pi/2} \tan x \cos x$
49. $\lim_{x\to \pi/2} \tan x \sin(2x)$
50. $\lim_{x\to 1^+} \frac{1}{\ln x} - \frac{1}{x-1}$
51. $\lim_{x\to 3^+} \frac{5}{x^2-9} - \frac{x}{x-3}$
52. $\lim_{x\to \infty} x\tan(1/x)$
53. $\lim_{x\to \infty} \frac{(\ln x)^3}{x}$
54. $\lim_{x\to 1} \frac{x^2+x-2}{\ln x}$
55. $\lim_{x\to 0^+} \frac{1 - x^x}{x}$

Solutions 6.8

1. $0/0, \infty/\infty, 0\cdot\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$
2. F
3. F
4. The base of an expression is approaching 1 while its power is growing without bound.
5. derivatives; limits
6. Answers will vary.
7. Answers will vary.
8. Answers will vary.
9. 3
10. $-5/3$
11. -1
12. $-\sqrt{2}/2$
13. 5
14. 0
15. $2/3$
16. a/b
17. ∞
18. $1/2$
19. 0
20. 0
21. 0
22. ∞
23. ∞
24. ∞
25. 0
26. 2
27. -2
28. 0
29. 0
30. 0
31. 0
32. 0
33. ∞
34. ∞
35. ∞
36. 0
37. 0
38. e
39. 1
40. 1
41. 1
42. 1
43. 1
44. 0
45. 1
46. 1
47. 1
48. 1
49. 2
50. $1/2$
51. $-\infty$
52. 1
53. 0
54. 3
55. $+\infty$

Chapter 7 Techniques of Integration

7.1A Method of Substitution Review

In applications, simple integrals like $\int \cos x \, dx$ are rare. It is more likely you will encounter integrals like $\int \cos(2\pi x)\, dx$ or $\int \cos(2.34x + 7.49)\, dx$. Fortunately these can often be worked with a slightly modified table of integrals. The idea of u is it can be any differentiable change of variable. ***Live math!***

Integral Table (Change of variable Form)

$$\int du = u + C$$

$$\int e^u \, du = e^u + C$$

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C$$

$$\int a^u \, du = \frac{a^u}{\ln a} + C$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \sin u \, du = -\cos u + C$$

$$\int \sec^2 u \, du = \tan u + C$$

$$\int \csc^2 u \, du = -\cot u + C$$

$$\int \sec u \tan u \, du = \sec u + C$$

$$\int \csc u \cot u \, du = -\csc u + C$$

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$$

$$\int \frac{du}{1+u^2} = \arctan u + C$$

Method of Substitution

$$\int f(g(x))\, g'(x)\, dx \underset{du = g'(x)\,dx}{\overset{u = g(x)}{=}} \int f(u)\, du$$

Proof: the integral is live mathematics.

Examples

$\int \sin^3 x \cos x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$= \int u^3 \, du$

$= \frac{u^4}{4} + C$

$= \frac{1}{4}\sin^4 x + C.$

$\int_0^4 \sqrt{2x+1}\, dx$

$u = 2x + 1$

$du = 2dx \implies dx = \frac{du}{2}$

$x = 0 \implies u = 2\cdot 0 + 1 = 1$

$x = 4 \implies u = 2\cdot 4 + 1 = 9$

$= \int_1^9 \sqrt{u}\, \frac{du}{2}$

$= \frac{1}{2}\int_1^9 u^{1/2}\, du$

$= \frac{1}{2}\, \frac{2}{3}\, u^{3/2} \Big|_1^9$

$= \frac{1}{3}(9 - 1)$

$= \frac{8}{3}$

It is rare for an integral we are trying to evaluate to be exactly one on our *Memory Integral List*, really our only way so far of evaluating integrals other than making the list longer. The *Method of Substitution* is by far the most common and important way to transform an integral into one on the list.

This method of evaluating integrals is so important we will spend two lectures on getting fluent at it.

The previous chapter we completed the basic calculus of most of the functions a well educated calculus user should know.

Most combinations of these functions are easy to differentiate. However, many are difficult to integrate. This chapter is devoted to exploring techniques of antidifferentiation. While not every function has an antiderivative in terms of elementary functions we can still find antiderivatives of a wide variety of functions. Nevertheless, many remain impossible to integrate. For these we will learn approximate integration; however, realistically this job is best done by computer.

7.1 A Substitution Readings

We motivate this section with an example. Let $f(x) = (x^2 + 3x - 5)^{10}$.
We can compute $f'(x)$ using the Chain Rule. It is:
$f'(x) = 10(x^2 + 3x - 5)^9 \cdot (2x + 3) = (20x + 30)(x^2 + 3x - 5)^9$.

Now consider this: What is $\int (20x + 30)(x^2 + 3x - 5)^9\, dx$? We have the answer in front of us;

$$\int (20x + 30)(x^2 + 3x - 5)^9\, dx = (x^2 + 3x - 5)^{10} + C.$$

How would we have evaluated this indefinite integral without starting with $f(x)$ as we did?

This section explores *integration by substitution.* It allows us to "undo the Chain Rule." Substitution allows us to evaluate the above integral without knowing the original function first.

The underlying principle is to rewrite a "complicated" integral of the form $\int f(x)\, dx$ as a not–so–complicated integral $\int h(u)\, du$. We'll formally establish later how this is done. First, consider again our introductory indefinite integral, $\int (20x + 30)(x^2 + 3x - 5)^9\, dx$. Arguably the most "complicated" part of the integrand is $(x^2 + 3x - 5)^9$. We wish to make this simpler; we do so through a substitution. Let $u = x^2 + 3x - 5$. Thus

$$(x^2 + 3x - 5)^9 = u^9.$$

We have established u as a function of x, so now consider the differential of u:

$$du = (2x+3)dx.$$

Keep in mind that $(2x+3)$ and dx are multiplied; the dx is not "just sitting there."

Return to the original integral and do some substitutions through algebra:

$$\begin{aligned}
\int (20x+30)(x^2+3x-5)^9\,dx &= \int 10(2x+3)(x^2+3x-5)^9\,dx \\
&= \int 10(\underbrace{x^2+3x-5}_{u})^9\,\underbrace{(2x+3)\,dx}_{du} \\
&= \int 10u^9\,du \\
&= u^{10}+C \quad \text{(replace } u \text{ with } x^2+3x-5\text{)} \\
&= (x^2+3x-5)^{10}+C
\end{aligned}$$

One might well look at this and think "I (sort of) followed how that worked, but I could never come up with that on my own," but the process is learnable. This section contains numerous examples through which the reader will gain understanding and mathematical maturity enabling them to regard substitution as a natural tool when evaluating integrals.

We stated before that integration by substitution "undoes" the Chain Rule. Specifically, let $F(x)$ and $g(x)$ be differentiable functions and consider the derivative of their composition:

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'(g(x))g'(x).$$

Thus

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

Integration by substitution works by recognizing the "inside" function $g(x)$ and replacing it with a variable. By setting $u = g(x)$, we can rewrite the derivative as

$$\frac{d}{dx}\Big(F\big(u\big)\Big) = F'(u)u'.$$

Since $du = g'(x)dx$, we can rewrite the above integral as

$$\int F'(g(x))g'(x)\,dx = \int F'(u)du = F(u) + C = F(g(x)) + C.$$

This concept is important so we restate it in the context of a theorem.

Theorem 7.1.1 Integration by Substitution

Let F and g be differentiable functions, where the range of g is an interval I contained in the domain of F. Then

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)dx$ and

$$\int F'(g(x))g'(x)\,dx = \int F'(u)\,du = F(u) + C = F(g(x)) + C.$$

The Method

Choose a u for which there is

(up to a constant) a du in the correct position.

The point of substitution is to make the integration step easy. Indeed, the step $\int F'(u)\,du = F(u) + C$ looks easy, as the antiderivative of the derivative of F is just F, plus a constant. The "work" involved is making the proper substitution. There is not a step–by–step process that one can memorize; rather, experience will be one's guide. To gain experience, we now embark on many examples.

Example 7.1.1 Integrating by substitution

Evaluate $\displaystyle\int x\sin(x^2+5)\,dx$.

We see a u
and except for a 2, a du.

Solution Knowing that substitution is related to the Chain Rule, we choose to let u be the "inside" function of $\sin(x^2+5)$*. (This is not *always* a good choice, but it is often the best place to start.)

Let $u = x^2 + 5$, hence $du = 2x\,dx$. The integrand has an $x\,dx$ term, but not a $2x\,dx$ term. (Recall that multiplication is commutative, so the x does not physically have to be next to dx for there to be an $x\,dx$ term.) We can divide both sides of the du expression by 2:

$$du = 2x\,dx \quad\Rightarrow\quad \frac{1}{2}du = x\,dx.$$

We can now substitute.

$$\int x\sin(x^2+5)\,dx = \int \sin(\underbrace{x^2+5}_{u})\,\underbrace{x\,dx}_{\frac{1}{2}du}$$
$$= \int \frac{1}{2}\sin u\,du$$

Eventually you can do these in your head with perhaps a little 'massaging' of the integrand.

$$\int x\sin(x^2+5)\,dx.$$

Thinking u = x^2 - 5

$$= \tfrac{1}{2}\int \sin(x^2+5)\,(2xdx)$$

$$= -\tfrac{1}{2}\cos(x^2+5) + C$$

After a while, you will be able to do easy ones completely in your head and immediately write down the answer.

* **alternate way of thinking that is productive and insightful:**

The Method

Choose a u for which there is

(up to a constant) a du in the correct position

ǿǻǘK ƴƵǘKǻƴƎ ƭŜŦǘ ǊŜY Ɨ-ǻƴǻƴg.

$$= -\frac{1}{2}\cos u + C \quad \text{(now replace } u \text{ with } x^2+5\text{)}$$
$$= -\frac{1}{2}\cos(x^2+5) + C.$$

Thus $\int x\sin(x^2+5)\,dx = -\frac{1}{2}\cos(x^2+5)+C$. We can check our work by evaluating the derivative of the right hand side.

Example 7.1.2 Integrating by substitution

Evaluate $\displaystyle\int \cos(5x)\,dx$.

Solution Again let u replace the "inside" function. Letting $u = 5x$, we have $du = 5dx$. Since our integrand does not have a $5dx$ term, we can divide the previous equation by 5 to obtain $\frac{1}{5}du = dx$. We can now substitute.

$$\int \cos(5x)\,dx = \int \cos(\underbrace{5x}_{u})\,\underbrace{dx}_{\frac{1}{5}du}$$

We see a u and except for a 5, a du.

$$= \int \frac{1}{5}\cos u\,du$$
$$= \frac{1}{5}\sin u + C$$
$$= \frac{1}{5}\sin(5x) + C.$$

We can again check our work through differentiation.

The previous example exhibited a common, and simple, type of substitution. The "inside" function was a linear function (in this case, $y = 5x$). When the inside function is linear, the resulting integration is very predictable, outlined here.

Key Idea 7.1.1 Substitution With A Linear Function

Consider $\int F'(ax+b)\,dx$, where $a \neq 0$ and b are constants. Letting $u = ax+b$ gives $du = a\cdot dx$, leading to the result

$$\int F'(ax+b)\,dx = \frac{1}{a}F(ax+b)+C.$$

Thus $\int \sin(7x-4)\,dx = -\frac{1}{7}\cos(7x-4)+C$. Our next example can use Key Idea 6.1.1, but we will only employ it after going through all of the steps.

Example 7.1.3 Integrating by substituting a linear function

Evaluate $\displaystyle\int \frac{7}{-3x+1}\,dx$.

Solution View the integrand as the composition of functions $f(g(x))$, where $f(x) = 7/x$ and $g(x) = -3x+1$. Employing our understanding of substitution, we let $u = -3x+1$, the inside function. Thus $du = -3dx$. The integrand lacks a -3; hence divide the previous equation by -3 to obtain $-du/3 = dx$. We can now evaluate the integral through substitution.

$$\begin{aligned}\int \frac{7}{-3x+1}\,dx &= \int \frac{7}{u}\frac{du}{-3}\\ &= \frac{-7}{3}\int\frac{du}{u}\\ &= \frac{-7}{3}\ln|u| + C\\ &= -\frac{7}{3}\ln|-3x+1| + C.\end{aligned}$$

Using Key Idea 6.1.1 is faster, recognizing that u is linear and $a = -3$. One may want to continue writing out all the steps until they are comfortable with this particular shortcut.

Not all integrals that benefit from substitution have a clear "inside" function. Several of the following examples will demonstrate ways in which this occurs.

Example 7.1.4 Integrating by substitution

Evaluate $\displaystyle\int \sin x\cos x\,dx$.

Solution There is not a composition of function here to exploit; rather, just a product of functions. Do not be afraid to experiment; when given an integral to evaluate, it is often beneficial to think "If I let u be *this*, then du must be *that* …" and see if this helps simplify the integral at all.

In this example, let's set $u = \sin x$. Then $du = \cos x\,dx$, which we have as part of the integrand! The substitution becomes very straightforward:

$$\begin{aligned}\int \sin x\cos x\,dx &= \int u\,du\\ &= \frac{1}{2}u^2 + C\\ &= \frac{1}{2}\sin^2 x + C.\end{aligned}$$

We see a u and exactly, a du.

One would do well to ask "What would happen if we let $u = \cos x$?" The result is just as easy to find, yet looks very different. The challenge to the reader is to evaluate the integral letting $u = \cos x$ and discover why the answer is the same, yet looks different.

Our examples so far have required "basic substitution." The next example demonstrates how substitutions can be made that often strike the new learner as being "nonstandard."

Example 7.1.5 Integrating by substitution variation

Evaluate $\displaystyle\int x\sqrt{x+3}\,dx$.

Solution Recognizing the composition of functions, set $u = x + 3$. Then $du = dx$, giving what seems initially to be a simple substitution. But at this stage, we have:

$$\int x\sqrt{x+3}\,dx = \int x\sqrt{u}\,du.$$

We cannot evaluate an integral that has both an x and an u in it. We need to convert the x to an expression involving just u.

Since we set $u = x+3$, we can also state that $u-3 = x$. Thus we can replace x in the integrand with $u - 3$. It will also be helpful to rewrite $\sqrt{u}$ as $u^{\frac{1}{2}}$.

$$\begin{aligned}\int x\sqrt{x+3}\,dx &= \int (u-3)u^{\frac{1}{2}}\,du\\ &= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right) du\\ &= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C\\ &= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.\end{aligned}$$

Checking your work is always a good idea. In this particular case, some algebra will be needed to make one's answer match the integrand in the original problem.

Example 7.1.6 Integrating by substitution

Evaluate $\displaystyle\int \frac{1}{x\ln x}\,dx$.

Solution This is another example where there does not seem to be an obvious composition of functions. The line of thinking used in Example 6.1.5 is useful here: choose something for u and consider what this implies du must be. If u can be chosen such that du also appears in the integrand, then we have chosen well.

Choosing $u = 1/x$ makes $du = -1/x^2\,dx$; that does not seem helpful. However, setting $u = \ln x$ makes $du = 1/x\,dx$, which is part of the integrand. Thus:

$$\begin{aligned}
\int \frac{1}{x\ln x}\,dx &= \int \underbrace{\frac{1}{\ln x}}_{u}\,\underbrace{\frac{1}{x}\,dx}_{du} \\
&= \int \frac{1}{u}\,du \\
&= \ln|u| + C \\
&= \ln|\ln x| + C.
\end{aligned}$$

The final answer is interesting; the natural log of the natural log. Take the derivative to confirm this answer is indeed correct.

Integrals Involving Trigonometric Functions Section 7.3 delves deeper into integrals of a variety of trigonometric func-tions; here we use substitution to establish a foundation that we will build upon.

The next three examples will help fill in some missing pieces of our antideriva-tive knowledge. We know the antiderivatives of the sine and cosine functions; what about the other standard functions tangent, cotangent, secant and cosecant? We discover these next.

Example 7.1.7 Integration by substitution: antiderivatives of $\tan x$

Evaluate $\displaystyle\int \tan x\,dx$.

Solution The previous paragraph established that we did not know the antiderivatives of tangent, hence we must assume that we have learned something in this section that can help us evaluate this indefinite integral.

Rewrite $\tan x$ as $\sin x/\cos x$. While the presence of a composition of functions may not be immediately obvious, recognize that $\cos x$ is "inside" the $1/x$ function. Therefore, we see if setting $u = \cos x$ returns usable results. We have that $du = -\sin x\, dx$, hence $-du = \sin x\, dx$. We can integrate:

$$\begin{aligned}
\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\
&= \int \frac{1}{\underbrace{\cos x}_{u}} \underbrace{\sin x\, dx}_{-du} \\
&= \int \frac{-1}{u}\, du \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C.
\end{aligned}$$

Some texts prefer to bring the -1 inside the logarithm as a power of $\cos x$, as in:

$$\begin{aligned}
-\ln|\cos x| + C &= \ln|(\cos x)^{-1}| + C \\
&= \ln\left|\frac{1}{\cos x}\right| + C \\
&= \ln|\sec x| + C.
\end{aligned}$$

Thus the result they give is $\int \tan x\, dx = \ln|\sec x| + C$. These two answers are equivalent.

Example 7.1.8 Integrating by substitution: antiderivatives of $\sec x$

Evaluate $\displaystyle\int \sec x\, dx$.

Solution This example employs a wonderful trick: multiply the integrand by "1" so that we see how to integrate more clearly. In this case, we write "1" as

$$1 = \frac{\sec x + \tan x}{\sec x + \tan x}.$$

This may seem like it came out of left* field, but it works beautifully. Consider:

$$\begin{aligned}
\int \sec x\, dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx.
\end{aligned}$$

***No disparaging of left handed persons intended.**

Now let $u = \sec x + \tan x$; this means $du = (\sec x \tan x + \sec^2 x)\,dx$, which is our numerator. Thus:

$$\begin{aligned} &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\sec x + \tan x| + C. \end{aligned}$$

We can use similar techniques to those used in Examples 7.1.7 and 7.1.8 to find antiderivatives of cot x and csc x (which the reader can explore in the exercises.) We summarize our results here.

Theorem 7.1.2 Antiderivatives of Trigonometric Functions

1. $\displaystyle\int \sin x\,dx = -\cos x + C$
2. $\displaystyle\int \cos x\,dx = \sin x + C$
3. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$
4. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
5. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$
6. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

We explore one more common trigonometric integral.

Example 7.1.9 Integration by substitution: powers of $\cos x$ and $\sin x$

Evaluate $\displaystyle\int \cos^2 x\,dx$.

SOLUTION We have a composition of functions as $\cos^2 x = \big(\cos x\big)^2$. However, setting $u = \cos x$ means $du = -\sin x\,dx$, which we do not have in the integral. Another technique is needed.

The process we'll employ is to use a Power Reducing formula for $\cos^2 x$ (perhaps consult the back of this text for this formula), which states

$$\cos^2 x = \frac{1 + \cos(2x)}{2}.$$

The right hand side of this equation is not difficult to integrate. We have:

$$\begin{aligned} \int \cos^2 x\,dx &= \int \frac{1 + \cos(2x)}{2}\,dx \\ &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\,dx. \end{aligned}$$

Now use Key Idea 7.1.1:

$$= \frac{1}{2}x + \frac{1}{2}\frac{\sin(2x)}{2} + C$$
$$= \frac{1}{2}x + \frac{\sin(2x)}{4} + C.$$

We'll make significant use of this power–reducing technique in future sections.

Simplifying the Integrand

It is common to be reluctant to manipulate the integrand of an integral; at first, our grasp of integration is tenuous and one may think that working with the integrand will improperly change the results. Integration by substitution works using a different logic: as long as *equality* is maintained, the integrand can be manipulated so that its *form* is easier to deal with. The next two examples demonstrate common ways in which using algebra first makes the integration easier to perform.

Example 7.1.10 Integration by substitution: simplifying first

Evaluate $\displaystyle\int \frac{x^3 + 4x^2 + 8x + 5}{x^2 + 2x + 1}\,dx.$

Solution One may try to start by setting u equal to either the numerator or denominator; in each instance, the result is not workable.

When dealing with rational functions (i.e., quotients made up of polynomial functions), it is an almost universal rule that everything works better when the degree of the numerator is less than the degree of the denominator. Hence we use polynomial division.

We skip the specifics of the steps, but note that when $x^2 + 2x + 1$ is divided into $x^3 + 4x^2 + 8x + 5$, it goes in $x + 2$ times with a remainder of $3x + 3$. Thus

$$\frac{x^3 + 4x^2 + 8x + 5}{x^2 + 2x + 1} = x + 2 + \frac{3x + 3}{x^2 + 2x + 1}.$$

Integrating $x + 2$ is simple. The fraction can be integrated by setting $u = x^2 + 2x + 1$, giving $du = (2x + 2)\,dx$. This is very similar to the numerator. Note that $du/2 = (x + 1)\,dx$ and then consider the following:

$$\int \frac{x^3 + 4x^2 + 8x + 5}{x^2 + 2x + 1}\,dx = \int \left(x + 2 + \frac{3x + 3}{x^2 + 2x + 1}\right)\,dx$$
$$= \int (x + 2)\,dx + \int \frac{3(x + 1)}{x^2 + 2x + 1}\,dx$$
$$= \frac{1}{2}x^2 + 2x + C_1 + \int \frac{3}{u}\frac{du}{2}$$
$$= \frac{1}{2}x^2 + 2x + C_1 + \frac{3}{2}\ln|u| + C_2$$
$$= \frac{1}{2}x^2 + 2x + \frac{3}{2}\ln|x^2 + 2x + 1| + C.$$

Yes, you should have learned long division of polynomials in high school. If you were tought synthetic division, feel free to forget it.

In some ways, we "lucked out" in that after dividing, substitution was able to be done. In later sections we'll develop techniques for handling rational functions where substitution is not directly feasible.

Example 7.1.11 Integration by alternate methods
Evaluate $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx$ with, and without, substitution.

Solution We already know how to integrate this particular example. Rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$ and simplify the fraction:

$$\frac{x^2+2x+3}{x^{1/2}} = x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}.$$

We can now integrate using the Power Rule:

$$\begin{aligned}\int \frac{x^2+2x+3}{x^{1/2}}\,dx &= \int \left(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}\right)\,dx \\ &= \frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + C\end{aligned}$$

This is a perfectly fine approach. We demonstrate how this can also be solved using substitution as its implementation is rather clever.

Let $u = \sqrt{x} = x^{\frac{1}{2}}$; therefore

$$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}\,dx \quad\Rightarrow\quad 2du = \frac{1}{\sqrt{x}}\,dx.$$

This gives us $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx = \int (x^2+2x+3)\cdot 2\,du$. What are we to do with the other x terms? Since $u = x^{\frac{1}{2}}$, $u^2 = x$, etc. We can then replace x^2 and x with appropriate powers of u. We thus have

$$\begin{aligned}\int \frac{x^2+2x+3}{\sqrt{x}}\,dx &= \int (x^2+2x+3)\cdot 2\,du \\ &= \int 2(u^4+2u^2+3)\,du \\ &= \frac{2}{5}u^5 + \frac{4}{3}u^3 + 6u + C \\ &= \frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + C,\end{aligned}$$

which is obviously the same answer we obtained before. In this situation, substitution is arguably more work than our other method. The fantastic thing is that it works. It demonstrates how flexible integration is.

Exercises 7.1 A

Terms and Concepts

1. Substitution "undoes" what derivative rule?

2. T/F: One can use algebra to rewrite the integrand of an integral to make it easier to evaluate.

Problems

In Exercises 3 – 14, evaluate the indefinite integral to develop an understanding of Substitution.

3. $\int 3x^2\left(x^3-5\right)^7\,dx$

4. $\int (2x-5)\left(x^2-5x+7\right)^3\,dx$

5. $\int x\left(x^2+1\right)^8\,dx$

6. $\int (12x+14)\left(3x^2+7x-1\right)^5\,dx$

7. $\int \frac{1}{2x+7}dx$

8. $\int \frac{1}{\sqrt{2x+3}}dx$

9. $\int \frac{x}{\sqrt{x+3}}dx$

10. $\int \frac{x^3-x}{\sqrt{x}}dx$

11. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

12. $\int \frac{x^4}{\sqrt{x^5+1}}dx$

13. $\int \frac{\frac{1}{x}+1}{x^2}dx$

14. $\int \frac{\ln(x)}{x}dx$

In Exercises 15 – 24, use Substitution to evaluate the indefinite integral involving trigonometric functions.

15. $\int \sin^2(x)\cos(x)dx$

16. $\int \cos^3(x)\sin(x)dx$

35. $\int \frac{\ln\left(x^3\right)}{x}dx$

Solutions 7.1 A

1. Chain Rule.
2. T

$\frac{1}{8}(x^3-5)^8+C$

3.
4. $\frac{1}{4}(x^2-5x+7)^4+C$
5. $\frac{1}{18}\left(x^2+1\right)^9+C$
6. $\frac{1}{3}(3x^2+7x-1)^6+C$
7. $\frac{1}{2}\ln|2x+7|+C$
8. $\sqrt{2x+3}+C$
9. $\frac{2}{3}(x+3)^{3/2}-6(x+3)^{1/2}+C=\frac{2}{3}(x-6)\sqrt{x+3}+C$
10. $\frac{2}{21}x^{3/2}\left(3x^2-7\right)+C$
11. $2e^{\sqrt{x}}+C$
12. $\frac{2\sqrt{x^5+1}}{5}+C$
13. $-\frac{1}{2x^2}-\frac{1}{x}+C$
14. $\frac{\ln^2(x)}{2}+C$

15. $\frac{\sin^3(x)}{3}+C$
16. $-\frac{\cos^4(x)}{4}+C$

35. $\frac{3}{2}(\ln x)^2+C$

7.1B Advanced Methods of Substitution. Definite Integrals

Integral Table*

$\int du = u + C$

$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$

$\int e^u\,du = e^u + C$

$\int a^u\,du = \frac{a^u}{\ln a} + C$

$\int \frac{du}{u} = \ln|u| + C$

$\int \cos u\,du = \sin u + C$

$\int \sin u\,du = -\cos u + C$

$\int \sec^2 u\,du = \tan u + C$

$\int \csc^2 u\,du = -\cot u + C$

$\int \sec u \tan u\,du = \sec u + C$

$\int \csc u \cot u\,du = -\csc u + C$

$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$

$\int \frac{du}{1+u^2} = \arctan u + C$

$$\int_a^b f(g(x))\,g'(x)\,dx \underset{du = g'(x)\,dx}{\overset{u=g(x)}{=}} \int_{g(a)}^{g(b)} f(u)\,du$$

Method of Substitution Definite Integral form

*** This is the basic integral list everyone should memorize and remember forever!**

First, for review, a few more indefinite integrals.

Substitution and Inverse Trigonometric Functions

When studying derivatives of inverse functions, we learned that

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}.$$

Applying the Chain Rule to this is not difficult; for instance,

$$\frac{d}{dx}\left(\tan^{-1} 5x\right) = \frac{5}{1+25x^2}.$$

We now explore how Substitution can be used to "undo" certain derivatives that are the result of the Chain Rule applied to Inverse Trigonometric functions. We begin with an example.

Example 6 7.1.12 Integrating by substitution: inverse trigonometric functions

Evaluate $\int \frac{1}{25+x^2}\,dx.$

Solution

gent function. Note:

$$\frac{1}{25+x^2} = \frac{1}{25(1+\frac{x^2}{25})}$$
$$= \frac{1}{25(1+\left(\frac{x}{5}\right)^2)}$$
$$= \frac{1}{25}\frac{1}{1+\left(\frac{x}{5}\right)^2}.$$

Thus
$$\int \frac{1}{25+x^2}\,dx = \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx.$$

This can be integrated using Substitution. Set $u = x/5$, hence $du = dx/5$ or $dx = 5du$. Thus

$$\begin{aligned}\int \frac{1}{25+x^2}\,dx &= \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx\\ &= \frac{1}{5}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{5}\tan^{-1} u + C\\ &= \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C\end{aligned}$$

Example 7.1.12 demonstrates a general technique that can be applied to other integrands that result in inverse trigonometric functions. The results are summarized here.

Theorem 7.1.3 Integrals Involving Inverse Trigonometric Functions

Let $a > 0$.

1. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
2. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

If you are going to be a big time user of calculus, it is worth memorizing these.

Let's practice using Theorem 7.1.3.

Example 7.1.13 Integrating by substitution: inverse trigonometric functions
Evaluate the given indefinite integrals.

$$1.\ \int \frac{1}{9+x^2}\,dx, \quad 2.\ \int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx \quad 3.\ \int \frac{1}{\sqrt{5-x^2}}\,dx.$$

Solution Each can be answered using a straightforward application of Theorem 7.1.3.

1. $\displaystyle\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3} + C$, as $a = 3$.

2. $\displaystyle\int \frac{1}{x\sqrt{x^2 - \frac{1}{100}}}\,dx = 10\sec^{-1} 10x + C$, as $a = \frac{1}{10}$.

3. $\displaystyle\int \frac{1}{\sqrt{5-x^2}} = \sin^{-1}\frac{x}{\sqrt{5}} + C$, as $a = \sqrt{5}$.

Most applications of Theorem 7.1.3 are not as straightforward. The next examples show some common integrals that can still be approached with this theorem.

Example 7.1.14 **Integrating by substitution: completing the square**
Evaluate $\displaystyle\int \frac{1}{x^2 - 4x + 13}\,dx$.

Solution Initially, this integral seems to have nothing in common with the integrals in Theorem 7.1.3. As it lacks a square root, it almost certainly is not related to arcsine or arcsecant. It is, however, related to the arctangent function.

We see this by *completing the square* in the denominator. We give a brief reminder of the process here.

Start with a quadratic with a leading coefficient of 1. It will have the form of x^2+bx+c. Take 1/2 of b, square it, and add/subtract it back into the expression. I.e.,

$$\begin{aligned} x^2 + bx + c &= \underbrace{x^2 + bx + \frac{b^2}{4}}_{(x+b/2)^2} - \frac{b^2}{4} + c \\ &= \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4} \end{aligned}$$

In our example, we take half of -4 and square it, getting 4. We add/subtract it into the denominator as follows:

$$\begin{aligned} \frac{1}{x^2 - 4x + 13} &= \frac{1}{\underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4 + 13} \\ &= \frac{1}{(x-2)^2 + 9} \end{aligned}$$

Yes, any serious calculus consumer is likely to do completing the square often.

We can now integrate this using the arctangent rule. Technically, we need to substitute first with $u = x - 2$, but we can employ Key Idea 7.1.1 instead. Thus we have

$$\int \frac{1}{x^2 - 4x + 13}\,dx = \int \frac{1}{(x-2)^2 + 9}\,dx = \frac{1}{3}\tan^{-1}\frac{x-2}{3} + C.$$

Example 7.1.15 Integrals requiring multiple methods

Evaluate $\displaystyle\int \frac{4-x}{\sqrt{16-x^2}}\,dx$.

Solution This integral requires two different methods to evaluate it. We get to those methods by splitting up the integral:

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = \int \frac{4}{\sqrt{16-x^2}}\,dx - \int \frac{x}{\sqrt{16-x^2}}\,dx.$$

The first integral is handled using a straightforward application of Theorem 7.1.3; the second integral is handled by substitution, with $u = 16 - x^2$. We handle each separately.

$\displaystyle\int \frac{4}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + C.$

$\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$: Set $u = 16 - x^2$, so $du = -2xdx$ and $xdx = -du/2$. We have

$$\begin{aligned}\int \frac{x}{\sqrt{16-x^2}}\,dx &= \int \frac{-du/2}{\sqrt{u}} \\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \\ &= -\sqrt{u} + C \\ &= -\sqrt{16-x^2} + C.\end{aligned}$$

Combining these together, we have

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + \sqrt{16-x^2} + C.$$

Substitution and Definite Integration

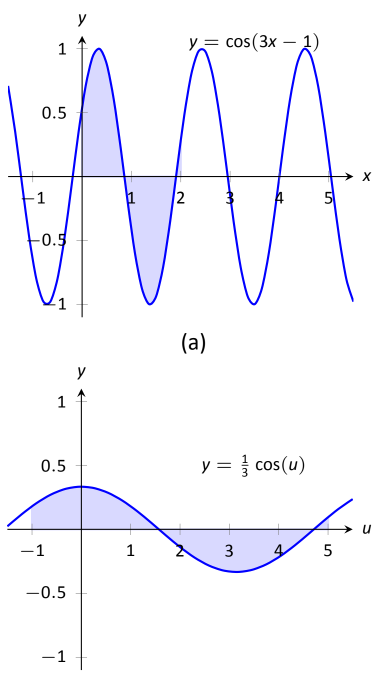
This section has focused on evaluating indefinite integrals as we are learning a new technique for finding antiderivatives. However, much of the time integration is used in the context of a definite integral. Definite integrals that require substitution can readily be calculated.

$$\int_a^b f(g(x))\, g'(x)\, dx \underset{\substack{x=a \Rightarrow u=g(a) \\ x=b \Rightarrow u=g(b)}}{\overset{u=g(x)}{=}} \int_{g(a)}^{g(b)} f(u)\, du$$

The integral is live mathematics.

> **Theorem 7.1.4 Substitution with Definite Integrals**
>
> Let F and g be differentiable functions, where the range of g is an interval I that is contained in the domain of F. Then
>
> $$\int_a^b F'\big(g(x)\big)g'(x)\, dx = \int_{g(a)}^{g(b)} F'(u)\, du.$$

In effect, Theorem 7.1.4 states that once you convert to integrating with re-spect to u, you do not need to switch back to evaluating with respect to x. A few examples will help one understand.



7.1.1: Graphing the areas defined by the definite integrals of Example 7.1.16.

Example 7.1.16 Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^2 \cos(3x-1)\, dx$ using Theorem 7.1.4.

Solution Observing the composition of functions, let $u = 3x - 1$, hence $du = 3dx$. As $3dx$ does not appear in the integrand, divide the latter equation by 3 to get $du/3 = dx$.

By setting $u = 3x - 1$, we are implicitly stating that $g(x) = 3x - 1$. Theorem 7.1.4 states that the new lower bound is $g(0) = -1$; the new upper bound is

$g(2) = 5$. We now evaluate the definite integral:

$$\begin{aligned}\int_0^2 \cos(3x-1)\,dx &= \int_{-1}^5 \cos u \frac{du}{3}\\ &= \frac{1}{3}\sin u\Big|_{-1}^5\\ &= \frac{1}{3}\big(\sin 5 - \sin(-1)\big) \doteq -0.039.\end{aligned}$$

Notice how once we converted the integral to be in terms of u, we never went back to using x.

The graphs in Figure 7.1.1 tell more of the story. In (a) the area defined by the original integrand is shaded, whereas in (b) the area defined by the new integrand is shaded. In this particular situation, the areas look very similar; the new region is "shorter" but "wider," giving the same area.

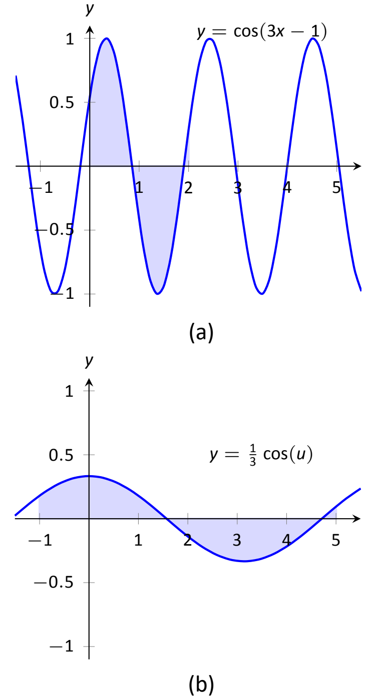


Figure 7.1.1: Graphing the areas defined by the definite integrals of Example 6.1.16.

Example 7.1.17 Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^{\pi/2} \sin x \cos x\,dx$ using Theorem 7.1.4.

Solution We saw the corresponding indefinite integral in Example 7.1.4. In that example we set $u = \sin x$ but stated that we could have let $u = \cos x$. For variety, we do the latter here.

Let $u = g(x) = \cos x$, giving $du = -\sin x\,dx$ and hence $\sin x\,dx = -du$. The new upper bound is $g(\pi/2) = 0$; the new lower bound is $g(0) = 1$. Note how the lower bound is actually larger than the upper bound now. We have

$$\begin{aligned}\int_0^{\pi/2} \sin x \cos x\,dx &= \int_1^0 -u\,du \quad \text{(switch bounds \& change sign)}\\ &= \int_0^1 u\,du\\ &= \frac{1}{2}u^2\Big|_0^1 = 1/2.\end{aligned}$$

In Figure 7.1.2 we have again graphed the two regions defined by our definite integrals. Unlike the previous example, they bear no resemblance to each other. However, Theorem 7.1.4 guarantees that they have the same area.

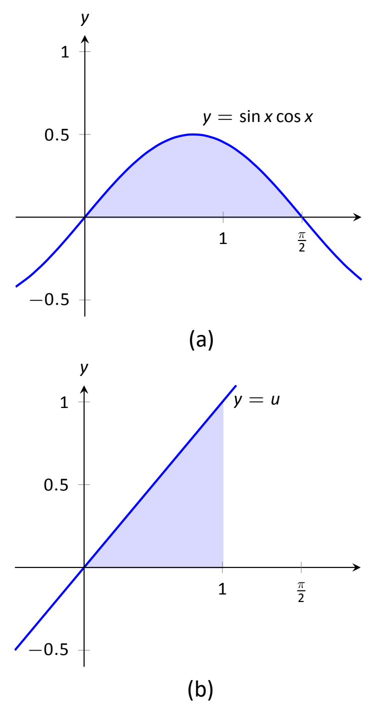


Figure 7.1.2: Graphing the areas de-fined by the definite integrals of Example 6.1.17.

Integration by substitution is a powerful and useful integration technique. The next section introduces another technique, called Integration by Parts. As substitution "undoes" the Chain Rule, integration by parts "undoes" the Product Rule. Together, these two techniques provide a strong foundation on which most other integration techniques are based.

Expert Method

Example **God's Method of Substitution:**

$\int (1+2x)^2 e^{x^2}\,dx$ Let $u = xe^{x^2}$

$du = (1\cdot e^{x^2} + x\cdot e^{x^2}\cdot 2x)dx$

$= (1+2x^2)\,e^{x^2}\,dx$

$= \int du$

$= u + C$

$= xe^{x^2} + C$

Exercises 7.1 B

17. $\int \cos(3-6x)dx$

18. $\int \sec^2(4-x)dx$

19. $\int \sec(2x)dx$

20. $\int \tan^2(x)\sec^2(x)dx$

21. $\int x\cos\left(x^2\right)\,dx$

22. $\int \tan^2(x)dx$

23. $\int \cot x\,dx$. Do not just refer to Theorem 6.1.2 for the answer; justify it through Substitution.

24. $\int \csc x\,dx$. Do not just refer to Theorem 6.1.2 for the answer; justify it through Substitution.

45. $\int \frac{14}{\sqrt{5-x^2}}dx$

46. $\int \frac{2}{x\sqrt{x^2-9}}dx$

47. $\int \frac{5}{\sqrt{x^4-16x^2}}dx$

In Exercises 79 – 86, evaluate the definite integral.

79. $\int_1^3 \frac{1}{x-5}dx$

80. $\int_2^6 x\sqrt{x-2}dx$

81. $\int_{-\pi/2}^{\pi/2} \sin^2 x\cos x\,dx$

82. $\int_0^1 2x(1-x^2)^4\,dx$

83. $\int_{-2}^{-1} (x+1)e^{x^2+2x+1}\,dx$

84. $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$

85. $\int_2^4 \frac{1}{x^2-6x+10}\,dx$

86. $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

Solutions 7.1 B

17. $-\frac{1}{6}\sin(3-6x)+C$
18. $-\tan(4-x)+C$
19. $\frac{1}{2}\ln|\sec(2x)+\tan(2x)|+C$
20. $\frac{\tan^3(x)}{3}+C$
21. $\frac{\sin\left(x^2\right)}{2}+C$
22. $\tan(x)-x+C$
23. The key is to rewrite $\cot x$ as $\cos x/\sin x$, and let $u=\sin x$.
24. The key is to multiply $\csc x$ by 1 in the form $(\csc x+\cot x)/(\csc x+\cot x)$.

79. $-\ln 2$
80. $352/15$
81. $2/3$
82. $1/5$
83. $(1-e)/2$
84. $\pi/2$
85. $\pi/2$
86. $\pi/6$

7.2 Integration by Parts

Here's a simple integral that we can't yet evaluate:

$$\int x \cos x \, dx.$$

It's a simple matter to take the derivative of the integrand using the Product Rule, but there is no Product Rule for integrals. However, this section introduces *Integration by Parts*, a method of integration that is based on the Product Rule for derivatives. It will enable us to evaluate this integral.

The Product Rule says that if u and v are functions of x, then $(uv)' = u'v + uv'$. For simplicity, we've written u for $u(x)$ and v for $v(x)$. Suppose we integrate both sides with respect to x. This gives

$$\int (uv)' \, dx = \int (u'v + uv') \, dx.$$

By the Fundamental Theorem of Calculus, the left side integrates to uv. The right side can be broken up into two integrals, and we have

$$uv = \int u'v \, dx + \int uv' \, dx.$$

Solving for the second integral we have

$$\int uv' \, dx = uv - \int u'v \, dx.$$

Using differential notation, we can write $du = u'(x)dx$ and $dv = v'(x)dx$ and the expression above can be written as follows:

$$\int u \, dv = uv - \int v \, du.$$

This is the Integration by Parts formula. For reference purposes, we state this in a theorem.

Theorem 7.2.1 Integration by Parts

Let u and v be differentiable functions of x on an interval I containing a and b. Then

$$\int u \, dv = uv - \int v \, du,$$

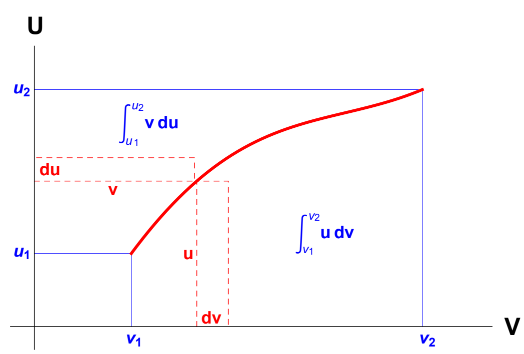
and

$$\int_{x=a}^{x=b} u \, dv = uv\Big|_a^b - \int_{x=a}^{x=b} v \, du.$$

Alternate Notation Derivaton

$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$	Product Rule
$d(uv) = v\,du + u\,dv$	Differential Form
$\int d(uv) = \int v\,du + \int u\,dv$	Integrating
$uv = \int v\,du + \int u\,dv$	
$\int u\,dv = uv - \int v\,du$	Rearranging

Proof by picture



Useful for:

1. **'unlikely products'**
2. **inverse functions**
3. **all else fails.**

Let's try an example to understand our new technique.

Example 7.2.1 Integrating using Integration by Parts

Evaluate $\displaystyle\int x\cos x\,dx$.

The following rule almost always works:

1. **Let dv = hardest part you can integrate**
2. **Let u = remaining part of the integral**

$$\begin{array}{ccc} dv = \cos x\,dx & \Rightarrow & v = \sin x \\ u = x & & du = dx \end{array}$$

Figure 7.2.1: Setting up Integration by Parts.

Now substitute all of this into the Integration by Parts formula, giving

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx.$$

We can then integrate $\sin x$ to get $-\cos x + C$ and overall our answer is

$$\int x\cos x\,dx = x\sin x + \cos x + C.$$

Note how the antiderivative contains a product, $x\sin x$. This product is what makes Integration by Parts necessary.

The example above demonstrates how Integration by Parts works in general. We try to identify u and dv in the integral we are given, and the key is that we usually want to choose u and dv so that du is simpler than u and v is hopefully not too much more complicated than dv. This will mean that the integral on the

In the example above, we chose $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ was simpler than u and $v = \sin x$ is no more complicated than dv. Therefore, instead of integrating $x\cos x\,dx$, we could integrate $\sin x\,dx$, which we knew how to do.

right side of the Integration by Parts formula, $v\,du$ will be simpler to integrate than the original integral $\int u\,dv$.

We now consider another example.

Example 7.2.2 Integrating using Integration by Parts

Evaluate $\int xe^x\,dx$.

SOLUTION The integrand contains an **A**lgebraic term (x) and an **E**xponential term (e^x). Our mnemonic suggests letting u be the algebraic term, so we choose $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$ as indicated by the tables below.

$$\begin{array}{ccc} dv = e^x\,dx & & v = e^x \\ & \Rightarrow & \\ u = x & & du = dx \end{array}$$

Figure 7.2.2: Setting up Integration by Parts.

We see du is simpler than u, while there is no change in going from dv to v. This is good. The Integration by Parts formula gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

The integral on the right is simple; our final answer is

$$\int xe^x\,dx = xe^x - e^x + C.$$

Note again how the antiderivatives contain a product term.

Example 7.2.3 Integrating using Integration by Parts

Evaluate $\int x^2 \cos x\,dx$.

SOLUTION The mnemonic suggests letting $u = x^2$ instead of the trigonometric function, hence $dv = \cos x\,dx$. Then $du = 2x\,dx$ and $v = \sin x$ as shown below.

$$\begin{array}{ccc} dv = \cos x\,dx & & v = \sin x \\ & \Rightarrow & \\ u = x^2 & & du = 2\text{x dx} \end{array}$$

Figure 7.2.3: Setting up Integration by Parts.

The Integration by Parts formula gives

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx.$$

At this point, the integral on the right is indeed simpler than the one we started with, but to evaluate it, we need to do Integration by Parts again. Here we choose $u = 2x$ and $dv = \sin x$ and fill in the rest below.

$$\begin{array}{rcr} dv = \sin x \, dx & \Rightarrow & v = -\cos x \\ u = 2x & & du = 2 \, dx \end{array}$$

Figure 7.2.4: Setting up Integration by Parts (again).

$$\int x^2 \cos x \, dx = x^2 \sin x - \left(-2x \cos x - \int -2 \cos x \, dx \right).$$

The integral all the way on the right is now something we can evaluate. It evaluates to $-2 \sin x$. Then going through and simplifying, being careful to keep all the signs straight, our answer is

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

Example 7.2.4 Integrating using Integration by Parts

Evaluate $\displaystyle\int e^x \cos x \, dx$.

Solution This is a classic problem. In this particular example, one can let dv be either $\cos x \, dx$ or e^x dx.

$$\begin{array}{rcr} dv = \cos x \, dx & \Rightarrow & v = \sin x \\ u = e^x & & du = e^x \, dx \end{array}$$

Figure 7.2.5: Setting up Integration by Parts.

Notice that du is no simpler than u, going against our general rule (but bear with us). The Integration by Parts formula yields

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The integral on the right is not much different than the one we started with, so it seems like we have gotten nowhere. Let's keep working and apply Integration by Parts to the new integral, using $u = e^x$ and $dv = \sin x\,dx$. This leads us to the following:

$$\begin{aligned} dv &= \sin x\,dx \\ u &= e^x \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} v &= -\cos x \\ du &= e^x\,dx \end{aligned}$$

Figure 7.2.6: Setting up Integration by Parts (again).

The Integration by Parts formula then gives:

$$\begin{aligned} \int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x\,dx\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx. \end{aligned}$$

'Oh, shoot'

It seems we are back right where we started, as the right hand side contains $\int e^x \cos x\,dx$. But this is actually a good thing.

Add $\displaystyle\int e^x \cos x\,dx$ to both sides. This gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

'Boot strapping!' In cowboy lore, boot strapping was the idea that you could lift yourself up into the air by pulling up on the straps at the top back of your cowboy boots.

Now divide both sides by 2:

$$\int e^x \cos x\,dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right).$$

Simplifying a little and adding the constant of integration, our answer is thus

$$\int e^x \cos x\,dx = \frac{1}{2}e^x\left(\sin x + \cos x\right) + C.$$

Example 7.2.5 Integrating using Integration by Parts: antiderivative of $\ln x$

Evaluate $\ln x\,dx$.

Solution One may have noticed that we have rules for integrating the familiar trigonometric functions and e^x, but we have not yet given a rule for integrating $\ln x$. That is because $\ln x$ can't easily be integrated with any of the rules we have learned up to this point. But we can find its antiderivative by a clever application of Integration by Parts. Set $u = \ln x$ and $dv = dx$. This is a good, sneaky trick to learn as it can help in other situations. This determines $du = (1/x)\,dx$ and $v = x$ as shown below.

$$\begin{aligned} dv &= dx \\ u &= \ln x \end{aligned} \quad \Rightarrow \quad \begin{aligned} v &= x \\ du &= 1/x\,dx \end{aligned}$$

Figure 7.2.7: Setting up Integration by Parts.

Putting this all together in the Integration by Parts formula, things work out very nicely:

$$\int \ln x\,dx = x\ln x - \int x\,\frac{1}{x}\,dx.$$

The new integral simplifies to $\int 1\,dx$, which is about as simple as things get. Its integral is $x + C$ and our answer is

$$\int \ln x\,dx = x\ln x - x + C.$$

Example 7.2.6 Integrating using Int. by Parts: antiderivative of $\arctan x$

Evaluate $\displaystyle\int \arctan x\,dx$.

Solution The same sneaky trick we used above works here. Let $dv = dx$ and $u = \arctan x$. Then v = x and $du = 1/(1+x^2)\,dx$.
The Integration by Parts formula gives

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

The integral on the right can be solved by substitution. Taking $u = 1 + x^2$, we get $du = 2x\,dx$. The integral then becomes

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du.$$

The integral on the right evaluates to $\ln|u| + C$, which becomes $\ln(1+x^2) + C$ (we can drop the absolute values as $1 + x^2$ is always postive). Therefore, the answer is

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

Substitution Before Integration

When taking derivatives, it was common to employ multiple rules (such as using both the Quotient and the Chain Rules). It should then come as no surprise that some integrals are best evaluated by combining integration techniques. In particular, here we illustrate making an "unusual" substitution first before using Integration by Parts.

Example 7.2.7 Integration by Parts after substitution
Evaluate $\int \cos(\ln x)\, dx$.

Solution The integrand contains a composition of functions, leading us to think Substitution would be beneficial. Letting $u = \ln x$, we have $du = 1/x\, dx$. This seems problematic, as we do not have a $1/x$ in the integrand. But consider:

$$du = \frac{1}{x}\, dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln x$, we can use inverse functions and conclude that $x = e^u$. Therefore we have that

$$\begin{aligned} dx &= x \cdot du \\ &= e^u\, du. \end{aligned}$$

We can thus replace $\ln x$ with u and dx with $e^u\, du$. Thus we rewrite our integral as

$$\int \cos(\ln x)\, dx = \int e^u \cos u\, du.$$

We evaluated this integral in Example 6.2.4. Using the result there, we have:

$$\begin{aligned} \int \cos(\ln x)\, dx &= \int e^u \cos u\, du \\ &= \frac{1}{2}e^u\big(\sin u + \cos u\big) + C \\ &= \frac{1}{2}e^{\ln x}\big(\sin(\ln x) + \cos(\ln x)\big) + C \\ &= \frac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big) + C. \end{aligned}$$

Definite Integrals and Integration By Parts

So far we have focused only on evaluating indefinite integrals. Of course, we can use Integration by Parts to evaluate definite integrals as well, as Theorem

7.2.1 states. We do so in the next example.

Example 7.2.8 Definite integration using Integration by Parts

Evaluate $\displaystyle\int_1^2 x^2 \ln x\, dx$.

SOLUTION Our mnemonic suggests letting $u = \ln x$, hence $dv = x^2\, dx$. We then get $du = (1/x)\, dx$ and $v = x^3/3$ as shown below.

$$\begin{array}{ccc} dv = x^2\, dx & & v = x^3/3 \\ & \Rightarrow & \\ u = \ln x & & du = 1/x\, dx \end{array}$$

Figure 6.2.8: Setting up Integration by Parts.

The Integration by Parts formula then gives

$$\begin{aligned}
\int_1^2 x^2 \ln x\, dx &= \frac{x^3}{3}\ln x\bigg|_1^2 - \int_1^2 \frac{x^3}{3}\frac{1}{x}\, dx \\
&= \frac{x^3}{3}\ln x\bigg|_1^2 - \int_1^2 \frac{x^2}{3}\, dx \\
&= \frac{x^3}{3}\ln x\bigg|_1^2 - \frac{x^3}{9}\bigg|_1^2 \\
&= \left(\frac{x^3}{3}\ln x - \frac{x^3}{9}\right)\bigg|_1^2 \\
&= \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(\frac{1}{3}\ln 1 - \frac{1}{9}\right) \\
&= \frac{8}{3}\ln 2 - \frac{7}{9} \\
&\approx 1.07.
\end{aligned}$$

In general, Integration by Parts is useful for integrating certain products of functions, like $\int xe^x\, dx$ or $\int x^3 \sin x\, dx$. It is also useful for integrals involving logarithms and inverse trigonometric functions.

As stated before, integration is generally more difficult than derivation. We are developing tools for handling a large array of integrals, and experience will tell us when one tool is preferable/necessary over another. For instance, consider the three similar–looking integrals

$$\int xe^x\, dx, \qquad \int xe^{x^2}\, dx \qquad \text{and} \qquad \int xe^{x^3}\, dx.$$

While the first is calculated easily with Integration by Parts, the second is best approached with Substitution. Taking things one step further, the third integral has no answer in terms of elementary functions, so none of the methods we learn in calculus will get us the exact answer.

Integration by Parts is a very useful method, second only to Substitution. In the following sections of this chapter, we continue to learn other integration techniques. The next section focuses on handling integrals containing trigonometric functions.

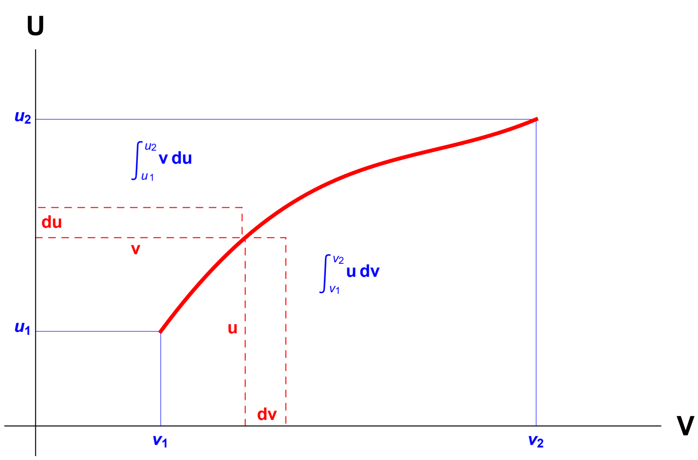
Let

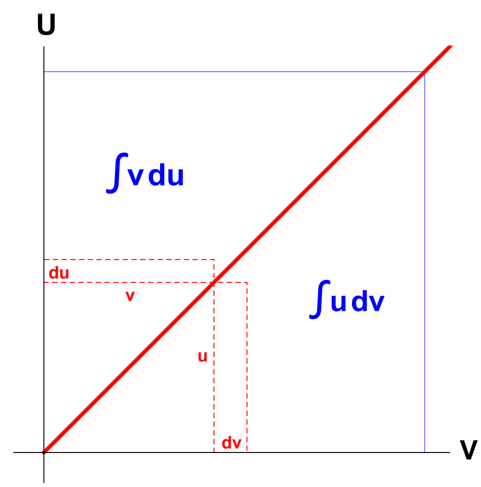
dv be the hardest part you can integrate

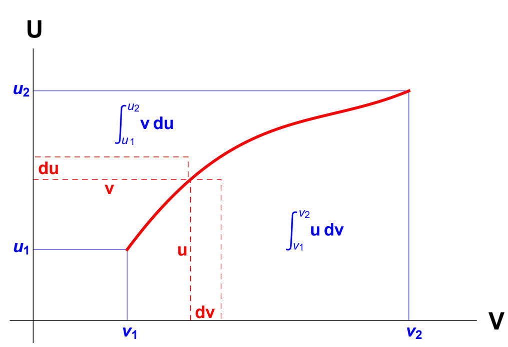
u = rest of the integrand

This rule seems always to work if integration by parts works.

Exercise **Attempts at graphic illustrations of Integration by Parts. Note how they try to work. Can you suggest improvements?**







Exercises 7.2

Terms and Concepts

1. T/F: Integration by Parts is useful in evaluating integrands that contain products of functions.

2. T/F: Integration by Parts can be thought of as the "opposite of the Chain Rule."

4. T/F: If the integral that results from Integration by Parts appears to also need Integration by Parts, then a mistake was made in the orginal choice of "u".

Problems

In Exercises 5 – 34, evaluate the given indefinite integral.

5. $\displaystyle\int x\sin x\,dx$

6. $\displaystyle\int xe^{-x}\,dx$

7. $\displaystyle\int x^2\sin x\,dx$

8. $\displaystyle\int x^3\sin x\,dx$

9. $\displaystyle\int xe^{x^2}\,dx$

10. $\displaystyle\int x^3e^x\,dx$

11. $\displaystyle\int xe^{-2x}\,dx$

12. $\displaystyle\int e^x\sin x\,dx$

13. $\displaystyle\int e^{2x}\cos x\,dx$

14. $\displaystyle\int e^{2x}\sin(3x)\,dx$

15. $\displaystyle\int e^{5x}\cos(5x)\,dx$

16. $\displaystyle\int \sin x\cos x\,dx$

17. $\displaystyle\int \sin^{-1}x\,dx$

18. $\displaystyle\int \tan^{-1}(2x)\,dx$

19. $\displaystyle\int x\tan^{-1}x\,dx$

20. $\displaystyle\int \sin^{-1}x\,dx$

21. $\displaystyle\int x\ln x\,dx$

22. $\displaystyle\int (x-2)\ln x\,dx$

23. $\displaystyle\int x\ln(x-1)\,dx$

24. $\displaystyle\int x\ln(x^2)\,dx$

25. $\displaystyle\int x^2\ln x\,dx$

26. $\displaystyle\int (\ln x)^2\,dx$

27. $\displaystyle\int (\ln(x+1))^2\,dx$

28. $\displaystyle\int x\sec^2x\,dx$

29. $\displaystyle\int x\csc^2x\,dx$

30. $\displaystyle\int x\sqrt{x-2}\,dx$

31. $\displaystyle\int x\sqrt{x^2-2}\,dx$

32. $\displaystyle\int \sec x\tan x\,dx$

33. $\displaystyle\int x\sec x\tan x\,dx$

34. $\displaystyle\int x\csc x\cot x\,dx$

In Exercises 35 – 40, evaluate the indefinite integral after first making a substitution.

35. $\displaystyle\int \sin(\ln x)\,dx$

36. $\displaystyle\int e^{2x}\cos\left(e^x\right)\,dx$

37. $\int \sin(\sqrt{x})\, dx$

38. $\int \ln(\sqrt{x})\, dx$

39. $\int e^{\sqrt{x}}\, dx$

40. $\int e^{\ln x}\, dx$

In Exercises 41 – 49, evaluate the definite integral. Note: the corresponding indefinite integrals appear in Exercises 5 – 13.

41. $\int_0^{\pi} x \sin x\, dx$

42. $\int_{-1}^{1} xe^{-x}\, dx$

43. $\int_{-\pi/4}^{\pi/4} x^2 \sin x\, dx$

44. $\int_{-\pi/2}^{\pi/2} x^3 \sin x\, dx$

45. $\int_0^{\sqrt{\ln 2}} xe^{x^2}\, dx$

46. $\int_0^1 x^3 e^x\, dx$

47. $\int_1^2 xe^{-2x}\, dx$

48. $\int_0^{\pi} e^x \sin x\, dx$

49. $\int_{-\pi/2}^{\pi/2} e^{2x} \cos x\, dx$

Solutions 7.2

10. $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$
11. $-\frac{1}{2}xe^{-2x} - \frac{e^{-2x}}{4} + C$
12. $1/2e^x(\sin x - \cos x) + C$
13. $1/5e^{2x}(\sin x + 2\cos x) + C$
14. $1/13e^{2x}(2\sin(3x) - 3\cos(3x)) + C$
15. $1/10e^{5x}(\sin(5x) + \cos(5x)) + C$
16. $-1/2\cos^2 x + C$
17. $\sqrt{1-x^2} + x\sin^{-1}(x) + C$
18. $x\tan^{-1}(2x) - \frac{1}{4}\ln\left|4x^2+1\right| + C$
19. $\frac{1}{2}x^2\tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\tan^{-1}(x) + C$
20. $\sqrt{1-x^2} + x\sin^{-1} x + C$
21. $\frac{1}{2}x^2\ln|x| - \frac{x^2}{4} + C$
22. $-\frac{x^2}{4} + \frac{1}{2}x^2\ln|x| + 2x - 2x\ln|x| + C$
23. $-\frac{x^2}{4} + \frac{1}{2}x^2\ln|x-1| - \frac{x}{2} - \frac{1}{2}\ln|x-1| + C$
24. $\frac{1}{2}x^2\ln\left(x^2\right) - \frac{x^2}{2} + C$
25. $\frac{1}{3}x^3\ln|x| - \frac{x^3}{9} + C$
26. $2x + x(\ln x)^2 - 2x\ln x + C$
27. $2(x+1) + (x+1)(\ln(x+1))^2 - 2(x+1)\ln(x+1) + C$
28. $x\tan(x) + \ln|\cos(x)| + C$
29. $\ln|\sin(x)| - x\cot(x) + C$
30. $\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$
31. $\frac{1}{3}(x^2-2)^{3/2} + C$
32. $\sec x + C$
33. $x\sec x - \ln|\sec x + \tan x| + C$
34. $-x\csc x - \ln|\csc x + \cot x| + C$
35. $1/2x\big(\sin(\ln x) - \cos(\ln x)\big) + C$
36. $\cos\left(e^x\right) + e^x\sin\left(e^x\right) + C$
37. $2\sin\left(\sqrt{x}\right) - 2\sqrt{x}\cos\left(\sqrt{x}\right) + C$
38. $\frac{1}{2}x\ln|x| - \frac{x}{2} + C$
39. $2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$
40. $1/2x^2 + C$
41. π
42. $-2/e$
43. 0
44. $\frac{3\pi^2}{2} - 12$
45. $1/2$
46. $6 - 2e$
47. $\frac{3}{4e^2} - \frac{5}{4e^4}$
48. $\frac{1}{2} + \frac{e^{\pi}}{2}$
49. $1/5\left(e^{\pi} + e^{-\pi}\right)$

7.3 Trigonometric Integrals

Functions involving trigonometric functions are useful as they are good at describing periodic behavior. This section describes several techniques for finding antiderivatives of certain combinations of trigonometric functions.

Integrals of the form $\int \sin^m x \cos^n x \, dx$

In learning the technique of Substitution, we saw the integral $\int \sin x \cos x \, dx$ in Example 6.1.4. The integration was not difficult, and one could easily evaluate the indefinite integral by letting $u = \sin x$ or by letting $u = \cos x$. This integral is easy since the power of both sine and cosine is 1.

We generalize this integral and consider integrals of the form $\int \sin^m x \cos^n x \, dx$, where m, n are nonnegative integers. Our strategy for evaluating these integrals is to use the identity $\cos^2 x + \sin^2 x = 1$ to convert high powers of one trigonometric function into the other, leaving a single sine or cosine term in the integrand. We summarize the general technique in the following Key Idea.

Semi-memorize these*

Key Idea 7.3.1 Integrals Involving Powers of Sine and Cosine

Consider $\displaystyle\int \sin^m x \cos^n x \, dx$, where m, n are nonnegative integers.

1. If m is odd, then $m = 2k+1$ for some integer k. Rewrite
$$\sin^m x = \sin^{2k+1} x = \sin^{2k} x \sin x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x.$$
Then
$$\int \sin^m x \cos^n x \, dx = \int (1-\cos^2 x)^k \sin x \cos^n x \, dx = -\int (1-u^2)^k u^n \, du,$$
where $u = \cos x$ and $du = -\sin x \, dx$.

2. If n is odd, then using substitutions similar to that outlined above we have
$$\int \sin^m x \cos^n x \, dx = \int u^m (1-u^2)^k \, du,$$
where $u = \sin x$ and $du = \cos x \, dx$.

3. If both m and n are even, use the power–reducing identities
$$\cos^2 x = \frac{1+\cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1-\cos(2x)}{2}$$
to reduce the degree of the integrand. Expand the result and apply the principles of this Key Idea again.

*** means you should be aware these methods exist in case you need them later.**

We practice applying Key Idea 7.3.1 in the next examples.

Example 7.3.1 Integrating powers of sine and cosine

Evaluate $\int \sin^5 x \cos^8 x\, dx$.

SOLUTION The power of the sine term is odd, so we rewrite $\sin^5 x$ as

$$\sin^5 x = \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1 - \cos^2 x)^2 \sin x.$$

Our integral is now $\int (1 - \cos^2 x)^2 \cos^8 x \sin x\, dx$. Let $u = \cos x$, hence $du = -\sin x\, dx$. Making the substitution and expanding the integrand gives

$$\int (1-\cos^2)^2 \cos^8 x \sin x\, dx = -\int (1-u^2)^2 u^8\, du = -\int \big(1-2u^2+u^4\big) u^8\, du = -\int \big(u^8-2u^{10}+u^{12}\big)\, du.$$

This final integral is not difficult to evaluate, giving

$$\begin{aligned} -\int \big(u^8 - 2u^{10} + u^{12}\big)\, du &= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C \\ &= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C. \end{aligned}$$

Example 7.3.2 Integrating powers of sine and cosine

Evaluate $\int \sin^5 x \cos^9 x\, dx$.

SOLUTION The powers of both the sine and cosine terms are odd, therefore we can apply the techniques of Key Idea 7.3.1 to either power. We choose to work with the power of the cosine term since the previous example used the sine term's power.

We rewrite $\cos^9 x$ as

$$\begin{aligned} \cos^9 x &= \cos^8 x \cos x \\ &= (\cos^2 x)^4 \cos x \\ &= (1 - \sin^2 x)^4 \cos x \\ &= (1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x) \cos x. \end{aligned}$$

We rewrite the integral as

$$\int \sin^5 x \cos^9 x\, dx = \int \sin^5 x \big(1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x\big) \cos x\, dx.$$

Now substitute and integrate, using $u = \sin x$ and $du = \cos x\,dx$.

$$\int \sin^5 x\big(1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x\big)\cos x\,dx =$$

$$\begin{aligned}\int u^5(1 - 4u^2 + 6u^4 - 4u^6 + u^8)\,du &= \int (u^5 - 4u^7 + 6u^9 - 4u^{11} + u^{13})\,du\\ &= \frac{1}{6}u^6 - \frac{1}{2}u^8 + \frac{3}{5}u^{10} - \frac{1}{3}u^{12} + \frac{1}{14}u^{14} + C\\ &= \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x + \ldots\\ &\quad - \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x + C.\end{aligned}$$

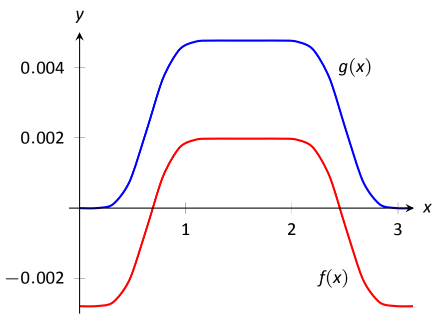


Figure 7.3.1: A plot of $f(x)$ and $g(x)$ from Example 7.3.2 and the Technology Note.

Technology Note: The work we are doing here can be a bit tedious, but the skills developed (problem solving, algebraic manipulation, etc.) are important. Nowadays problems of this sort are often solved using a computer algebra system. The powerful program *Mathematica*® integrates $\int \sin^5 x \cos^9 x\,dx$ as

$$f(x) = -\frac{45\cos(2x)}{16384} - \frac{5\cos(4x)}{8192} + \frac{19\cos(6x)}{49152} + \frac{\cos(8x)}{4096} - \frac{\cos(10x)}{81920} - \frac{\cos(12x)}{24576} - \frac{\cos(14x)}{114688},$$

which clearly has a different form than our answer in Example 6.3.2, which is

$$g(x) = \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x - \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x.$$

Figure 7.3.1 shows a graph of f and g; they are clearly not equal, but they differ *only by a constant*. That is $g(x) = f(x) + C$ for some constant C. So we have two different antiderivatives of the same function, meaning both answers are correct.

Example 7.3.3 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \cos^4 x \sin^2 x\,dx$.

Solution The powers of sine and cosine are both even, so we employ the power–reducing formulas and algebra as follows.

$$\begin{aligned}\int \cos^4 x\sin^2 x\,dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2\left(\frac{1-\cos(2x)}{2}\right)\,dx\\ &= \int \frac{1 + 2\cos(2x) + \cos^2(2x)}{4}\cdot\frac{1-\cos(2x)}{2}\,dx\\ &= \int \frac{1}{8}\big(1 + \cos(2x) - \cos^2(2x) - \cos^3(2x)\big)\,dx\end{aligned}$$

The $\cos(2x)$ term is easy to integrate, especially with Key Idea 7.1.1. The $\cos^2(2x)$ term is another trigonometric integral with an even power, requiring the power–reducing formula again. The $\cos^3(2x)$ term is a cosine function with an odd power, requiring a substitution as done before. We integrate each in turn below.

$$\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C.$$

$$\int \cos^2(2x)\,dx = \int \frac{1+\cos(4x)}{2}\,dx = \frac{1}{2}\big(x + \frac{1}{4}\sin(4x)\big) + C.$$

Finally, we rewrite $\cos^3(2x)$ as

$$\cos^3(2x) = \cos^2(2x)\cos(2x) = \big(1-\sin^2(2x)\big)\cos(2x).$$

Letting $u = \sin(2x)$, we have $du = 2\cos(2x)\,dx$, hence

$$\begin{aligned}\int \cos^3(2x)\,dx &= \int \big(1-\sin^2(2x)\big)\cos(2x)\,dx\\ &= \int \frac{1}{2}(1-u^2)\,du\\ &= \frac{1}{2}\Big(u - \frac{1}{3}u^3\Big) + C\\ &= \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big) + C\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}\int \cos^4 x\sin^2 x\,dx &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\,dx\\ &= \frac{1}{8}\Big[x + \frac{1}{2}\sin(2x) - \frac{1}{2}\big(x+\frac{1}{4}\sin(4x)\big) - \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big)\Big] + C\\ &= \frac{1}{8}\Big[\frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\Big] + C\end{aligned}$$

The process above was a bit long and tedious, but being able to work a problem such as this from start to finish is important.

Integrals of the form $\int \sin(mx)\sin(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$, and $\int \sin(mx)\cos(nx)\,dx$.

Functions that contain products of sines and cosines of differing periods are important in many applications including the analysis of sound waves. Integrals of the form

$$\int \sin(mx)\sin(nx)\,dx, \quad \int \cos(mx)\cos(nx)\,dx \quad \text{and} \quad \int \sin(mx)\cos(nx)\,dx$$

are best approached by first applying the Product to Sum Formulas found in the back cover of this text, namely

Be aware of these

$$\sin(mx)\sin(nx) = \frac{1}{2}\Big[\cos\big((m-n)x\big) - \cos\big((m+n)x\big)\Big]$$
$$\cos(mx)\cos(nx) = \frac{1}{2}\Big[\cos\big((m-n)x\big) + \cos\big((m+n)x\big)\Big]$$
$$\sin(mx)\cos(nx) = \frac{1}{2}\Big[\sin\big((m-n)x\big) + \sin\big((m+n)x\big)\Big]$$

Example 7.3.4 **Integrating products of $\sin(mx)$ and $\cos(nx)$**

Evaluate $\displaystyle\int \sin(5x)\cos(2x)\,dx$.

Solution The application of the formula and subsequent integration are straightforward:

$$\begin{aligned}\int \sin(5x)\cos(2x)\,dx &= \int \frac{1}{2}\Big[\sin(3x)+\sin(7x)\Big]\,dx\\ &= -\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C\end{aligned}$$

Integrals of the form $\displaystyle\int \tan^m x\sec^n x\,dx$.

When evaluating integrals of the form $\int \sin^m x\cos^n x\,dx$, the Pythagorean Theorem allowed us to convert even powers of sine into even powers of cosine, and vise–versa. If, for instance, the power of sine was odd, we pulled out one $\sin x$ and converted the remaining even power of $\sin x$ into a function using powers of $\cos x$, leading to an easy substitution.

The same basic strategy applies to integrals of the form $\int \tan^m x\sec^n x\,dx$, albeit a bit more nuanced. The following three facts will prove useful:

- $\frac{d}{dx}(\tan x) = \sec^2 x$,
- $\frac{d}{dx}(\sec x) = \sec x\tan x$, and
- $1+\tan^2 x = \sec^2 x$ (the Pythagorean Identity).

If the integrand can be manipulated to separate a $\sec^2 x$ term with the remaining secant power even, or if a $\sec x\tan x$ term can be separated with the remaining $\tan x$ power even, the Pythagorean Theorem can be employed, leading to a simple substitution. This strategy is outlined in the following Key Idea.

Key Idea 7.3.2 Integrals Involving Powers of Tangent and Secant

Consider $\displaystyle\int \tan^m x \sec^n x\, dx$, where m, n are nonnegative integers.

1. If n is even, then $n = 2k$ for some integer k. Rewrite $\sec^n x$ as

   $$\sec^n x = \sec^{2k} x = \sec^{2k-2} x \sec^2 x = (1 + \tan^2 x)^{k-1} \sec^2 x.$$

   Then

   $$\int \tan^m x \sec^n x\, dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x\, dx = \int u^m (1 + u^2)^{k-1}\, du,$$

   where $u = \tan x$ and $du = \sec^2 x\, dx$.

2. If m is odd, then $m = 2k + 1$ for some integer k. Rewrite $\tan^m x \sec^n x$ as

   $$\tan^m x \sec^n x = \tan^{2k+1} x \sec^n x = \tan^{2k} x \sec^{n-1} x \sec x \tan x = (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x.$$

   Then

   $$\int \tan^m x \sec^n x\, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\, dx = \int (u^2 - 1)^k u^{n-1}\, du,$$

   where $u = \sec x$ and $du = \sec x \tan x\, dx$.

3. If n is odd and m is even, then $m = 2k$ for some integer k. Convert $\tan^m x$ to $(\sec^2 x - 1)^k$. Expand the new integrand and use Integration By Parts, with $dv = \sec^2 x\, dx$.

4. If m is even and $n = 0$, rewrite $\tan^m x$ as

   $$\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x (\sec^2 x - 1) = \tan^{m-2} \sec^2 x - \tan^{m-2} x.$$

   So

   $$\int \tan^m x\, dx = \underbrace{\int \tan^{m-2} \sec^2 x\, dx}_{\text{apply rule \#1}} - \underbrace{\int \tan^{m-2} x\, dx}_{\text{apply rule \#4 again}}.$$

Semi-memorize these methods

The techniques described in items 1 and 2 of Key Idea 7.3.2 are relatively straightforward, but the techniques in items 3 and 4 can be rather tedious. A few examples will help with these methods.

Example 7.3.5 Integrating powers of tangent and secant
Evaluate $\displaystyle\int \tan^2 x \sec^6 x\, dx$.

Solution Since the power of secant is even, we use rule #1 from Key Idea 7.3.2 and pull out a $\sec^2 x$ in the integrand. We convert the remaining powers of secant into powers of tangent.

$$\begin{aligned}\int \tan^2 x \sec^6 x\, dx &= \int \tan^2 x \sec^4 x \sec^2 x\, dx \\ &= \int \tan^2 x \big(1+\tan^2 x\big)^2 \sec^2 x\, dx\end{aligned}$$

Now substitute, with $u = \tan x$, with $du = \sec^2 x\, dx$.

$$= \int u^2 \big(1+u^2\big)^2\, du$$

We leave the integration and subsequent substitution to the reader. The final answer is

$$= \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C.$$

Example 7.3.6 Integrating powers of tangent and secant
Evaluate $\displaystyle\int \sec^3 x\, dx$.

Solution We apply rule #3 from Key Idea 7.3.2 as the power of secant is odd and the power of tangent is even (0 is an even number). We use Integration by Parts; the rule suggests letting $dv = \sec^2 x\, dx$, meaning that $u = \sec x$.

$$\begin{array}{rcl} dv = \sec^2 x\, dx & & v = \tan x \\ & \Rightarrow & \\ u = \sec x & & du = \sec x \tan x\, dx \end{array}$$

Figure 7.3.2: Setting up Integration by Parts.

Employing Integration by Parts, we have

$$\begin{aligned}\int \sec^3 x\, dx &= \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x\, dx}_{dv} \\ &= \sec x \tan x - \int \sec x \tan^2 x\, dx.\end{aligned}$$

This new integral also requires applying rule #3 of Key Idea 7.3.2:

$$
\begin{aligned}
&= \sec x \tan x - \int \sec x\big(\sec^2 x - 1\big)\, dx \\
&= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx \\
&= \sec x \tan x - \int \sec^3 x\, dx + \ln|\sec x + \tan x|
\end{aligned}
$$

In previous applications of Integration by Parts, we have seen where the original integral has reappeared in our work. We resolve this by adding $\int \sec^3 x\, dx$ to both sides, giving:

$$
\begin{aligned}
2\int \sec^3 x\, dx &= \sec x \tan x + \ln|\sec x + \tan x| \\
\int \sec^3 x\, dx &= \frac{1}{2}\Big(\sec x \tan x + \ln|\sec x + \tan x|\Big) + C
\end{aligned}
$$

We give one more example.

Example 7.3.7 Integrating powers of tangent and secant

Evaluate $\displaystyle\int \tan^6 x\, dx$.

Solution We employ rule #4 of Key Idea 7.3.2.

$$
\begin{aligned}
\int \tan^6 x\, dx &= \int \tan^4 x \tan^2 x\, dx \\
&= \int \tan^4 x\big(\sec^2 x - 1\big)\, dx \\
&= \int \tan^4 x \sec^2 x\, dx - \int \tan^4 x\, dx
\end{aligned}
$$

Integrate the first integral with substitution, $u = \tan x$; integrate the second by employing rule #4 again.

$$
\begin{aligned}
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \tan^2 x\, dx \\
&= \frac{1}{5}\tan^5 x - \int \tan^2 x\big(\sec^2 x - 1\big)\, dx \\
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \sec^2 x\, dx + \int \tan^2 x\, dx
\end{aligned}
$$

Again, use substitution for the first integral and rule #4 for the second.

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \left(\sec^2 x - 1\right)\, dx$$
$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.$$

These latter examples were admittedly long, with repeated applications of the same rule. Try to not be overwhelmed by the length of the problem, but rather admire how robust this solution method is. A trigonometric function of a high power can be systematically reduced to trigonometric functions of lower powers until all antiderivatives can be computed.

The next section introduces an integration technique known as Trigonometric Substitution, a clever combination of Substitution and the Pythagorean Theorem.

Integral Table (Change of variable Form)

$$\int du = u + C$$

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C$$

$$\int e^u\, du = e^u + C$$

$$\int a^u\, du = \frac{a^u}{\ln a} + C$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int \cos u\, du = \sin u + C \qquad \int \sin u\, du = -\cos u + C$$

$$\int \sec^2 u\, du = \tan u + C \qquad \int \csc^2 u\, du = -\cot u + C$$

$$\int \sec u \tan u\, du = \sec u + C \qquad \int \csc u \cot u\, du = -\csc u + C$$

$$\int \tan u\, du = \ln(\sec u) + C \qquad \int \cot u\, du = \ln(\cos u) + C$$

$$\int \sec u\, du = \ln|\sec u + \tan u| + C \qquad \int \csc u\, du = \ln|\csc u - \cot u| + C$$

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C \qquad \int \frac{du}{1+u^2} = \arctan u + C$$

Method of Substitution

$$\int f(g(x))\, g'(x)\, dx \underset{du = g'(x)\,dx}{\overset{u = g(x)}{=}} \int f(u)\, du$$

$$\int_a^b f(g(x))\, g'(x)\, dx \underset{du = g'(x)\,dx}{\overset{u = g(x)}{=}} \int_{g(a)}^{g(b)} f(u)\, du$$

Proof: the integrals are live mathematics.

Exercises 7.3

Terms and Concepts

1. T/F: $\int \sin^2 x \cos^2 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are even.

2. T/F: $\int \sin^3 x \cos^3 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are odd.

3. T/F: This section addresses how to evaluate indefinite integrals such as $\int \sin^5 x \tan^3 x \, dx$.

4. T/F: Sometimes computer programs evaluate integrals involving trigonometric functions differently than one would using the techniques of this section. When this is the case, the techniques of this section have failed and one should only trust the answer given by the computer.

Problems

In Exercises 5 – 28, evaluate the indefinite integral.

5. $\int \sin x \cos^4 x \, dx$

6. $\int \sin^3 x \cos x \, dx$

7. $\int \sin^3 x \cos^2 x \, dx$

8. $\int \sin^3 x \cos^3 x \, dx$

9. $\int \sin^6 x \cos^5 x \, dx$

10. $\int \sin^2 x \cos^7 x \, dx$

11. $\int \sin^2 x \cos^2 x \, dx$

12. $\int \sin x \cos x \, dx$

13. $\int \sin(5x) \cos(3x) \, dx$

14. $\int \sin(x) \cos(2x) \, dx$

15. $\int \sin(3x) \sin(7x) \, dx$

16. $\int \sin(\pi x) \sin(2\pi x) \, dx$

17. $\int \cos(x) \cos(2x) \, dx$

18. $\int \cos\left(\frac{\pi}{2}x\right) \cos(\pi x) \, dx$

19. $\int \tan^4 x \sec^2 x \, dx$

20. $\int \tan^2 x \sec^4 x \, dx$

21. $\int \tan^3 x \sec^4 x \, dx$

22. $\int \tan^3 x \sec^2 x \, dx$

23. $\int \tan^3 x \sec^3 x \, dx$

24. $\int \tan^5 x \sec^5 x \, dx$

25. $\int \tan^4 x \, dx$

26. $\int \sec^5 x \, dx$

27. $\int \tan^2 x \sec x \, dx$

28. $\int \tan^2 x \sec^3 x \, dx$

In Exercises 29 – 35, evaluate the definite integral. Note: the corresponding indefinite integrals appear in the previous set.

29. $\int_0^{\pi} \sin x \cos^4 x \, dx$

30. $\int_{-\pi}^{\pi} \sin^3 x \cos x \, dx$

31. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^7 x \, dx$

32. $\int_0^{\pi/2} \sin(5x) \cos(3x) \, dx$

33. $\int_{-\pi/2}^{\pi/2} \cos(x) \cos(2x) \, dx$

34. $\int_0^{\pi/4} \tan^4 x \sec^2 x \, dx$

35. $\int_{-\pi/4}^{\pi/4} \tan^2 x \sec^4 x \, dx$

Solutions 7.3

1. F
2. F
3. F
4. F
5. $-\frac{1}{5}\cos^5(x)+C$
6. $\frac{1}{4}\sin^4(x)+C$
7. $\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+C$
8. $\frac{1}{6}\cos^6 x-\frac{1}{4}\cos^4 x+C$
9. $\frac{1}{11}\sin^{11} x-\frac{2}{9}\sin^9 x+\frac{1}{7}\sin^7 x+C$
10. $-\frac{1}{9}\sin^9(x)+\frac{3\sin^7(x)}{7}-\frac{3\sin^5(x)}{5}+\frac{\sin^3(x)}{3}+C$
11. $\frac{x}{8}-\frac{1}{32}\sin(4x)+C$
12. $\frac{1}{2}\sin^2 x+C$ or $-\frac{1}{2}\cos^2 x+C$, depending on the choice of substitution
13. $\frac{1}{2}\left(-\frac{1}{8}\cos(8x)-\frac{1}{2}\cos(2x)\right)+C$
14. $\frac{1}{2}\left(-\frac{1}{3}\cos(3x)+\cos(-x)\right)+C$
15. $\frac{1}{2}\left(\frac{1}{4}\sin(4x)-\frac{1}{10}\sin(10x)\right)+C$
16. $\frac{1}{2}\left(\frac{1}{\pi}\sin(\pi x)-\frac{1}{3\pi}\sin(3\pi x)\right)+C$
17. $\frac{1}{2}\left(\sin(x)+\frac{1}{3}\sin(3x)\right)+C$
18. $\frac{1}{\pi}\sin(\frac{\pi}{2}x)+\frac{1}{3\pi}\sin(\pi x)+C$
19. $\frac{\tan^5(x)}{5}+C$
20. $\frac{1}{5}\tan^5 x+\frac{1}{3}\tan^3 x+C$
21. $\frac{\tan^6(x)}{6}+\frac{\tan^4(x)}{4}+C$
22. $\frac{\tan^4(x)}{4}+C$
23. $\frac{\sec^5(x)}{5}-\frac{\sec^3(x)}{3}+C$
24. $\frac{\sec^9(x)}{9}-\frac{2\sec^7(x)}{7}+\frac{\sec^5(x)}{5}+C$
25. $\frac{1}{3}\tan^3 x-\tan x+x+C$
26. $\frac{1}{4}\tan x\sec^3 x+\frac{3}{8}\left(\sec x\tan x+\ln|\sec x+\tan x|\right)+C$
27. $\frac{1}{2}\left(\sec x\tan x-\ln|\sec x+\tan x|\right)+C$
28. $\frac{1}{4}\tan x\sec^3 x-\frac{1}{8}\left(\sec x\tan x+\ln|\sec x+\tan x|\right)+C$
29. $\frac{2}{5}$
30. 0
31. $32/315$
32. $1/2$
33. $2/3$
34. $1/5$
35. $16/15$

7.4 Trigonometric Substitution

In Section 5.2 we defined the definite integral as the "signed area under the curve." In that section we had not yet learned the Fundamental Theorem of Calculus, so we only evaluated special definite integrals which described nice, geometric shapes. For instance, we were able to evaluate

$$\int_{-3}^{3} \sqrt{9- x^2}\, dx = \frac{9}{2}\pi \qquad (7.1)$$

as we recognized that $f(x) = \sqrt{9 - x^2}$ described the upper half of a circle with radius 3.

We have since learned a number of integration techniques, including Substitution and Integration by Parts, yet we are still unable to evaluate the above integral without resorting to a geometric interpretation. This section introduces Trigonometric Substitution, a method of integration that fills this gap in our integration skill. This technique works on the same principle as Substitution as found in Section 6.1, though it can feel "backward." In Section 6.1, we set $u = f(x)$, for some function f, and replaced $f(x)$ with u. In this section, we will set $x = f(\theta)$, where f is a trigonometric function, then replace x with $f(\theta)$.

We start by demonstrating this method in evaluating the integral in Equation (7.1). After the example, we will generalize the method and give more examples.

Example 7.4.1 Using Trigonometric Substitution

Evaluate $\displaystyle\int_{-3}^{3} \sqrt{9 - x^2}\, dx$.

Solution We begin by noting that $9\sin^2\theta + 9\cos^2\theta = 9$, and hence $9\cos^2\theta = 9 - 9\sin^2\theta$. If we let $x = 3\sin\theta$, then $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$.

Setting $x = 3\sin\theta$ gives $dx = 3\cos\theta\, d\theta$. We are almost ready to substitute. We also wish to change our bounds of integration. The bound $x = -3$ corresponds to $\theta = -\pi/2$ (for when $\theta = -\pi/2$, $x = 3\sin\theta = -3$). Likewise, the bound of $x = 3$ is replaced by the bound $\theta = \pi/2$. Thus

$$\begin{aligned}
\int_{-3}^{3} \sqrt{9 - x^2}\, dx &= \int_{-\pi/2}^{\pi/2} \sqrt{9 - 9\sin^2\theta}(3\cos\theta)\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3\sqrt{9\cos^2\theta}\cos\theta\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3|3\cos\theta|\cos\theta\, d\theta.
\end{aligned}$$

On $[-\pi/2, \pi/2]$, $\cos\theta$ is always positive, so we can drop the absolute value bars, then employ a power–reducing formula:

$$
\begin{aligned}
&= \int_{-\pi/2}^{\pi/2} 9\cos^2\theta \, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{9}{2}\big(1+\cos(2\theta)\big) \, d\theta \\
&= \frac{9}{2}\big(\theta + \frac{1}{2}\sin(2\theta)\big)\Big|_{-\pi/2}^{\pi/2} = \frac{9}{2}\pi.
\end{aligned}
$$

This matches our answer from before.

We now describe in detail Trigonometric Substitution. This method excels when dealing with integrands that contain $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$. The following Key Idea outlines the procedure for each case, followed by more examples. Each right triangle acts as a reference to help us understand the relationships between x and θ.

Key Idea 7.4.1 Trigonometric Substitution

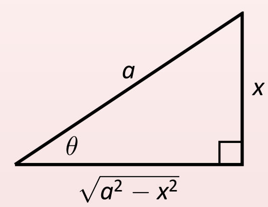
(a) For integrands containing $\sqrt{a^2-x^2}$:

Let $x = a\sin\theta$, $\quad dx = a\cos\theta\, d\theta$

Thus $\theta = \sin^{-1}(x/a)$, for $-\pi/2 \le \theta \le \pi/2$.

On this interval, $\cos\theta \ge 0$, so

$\sqrt{a^2-x^2} = a\cos\theta$



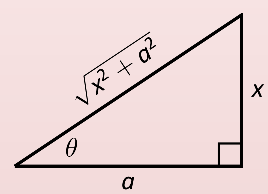
(b) For integrands containing $\sqrt{x^2+a^2}$:

Let $x = a\tan\theta$, $\quad dx = a\sec^2\theta\, d\theta$

Thus $\theta = \tan^{-1}(x/a)$, for $-\pi/2 < \theta < \pi/2$.

On this interval, $\sec\theta > 0$, so

$\sqrt{x^2+a^2} = a\sec\theta$



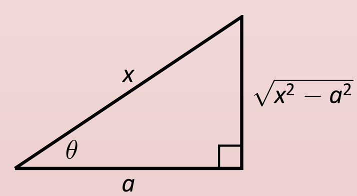
(c) For integrands containing $\sqrt{x^2-a^2}$:

Let $x = a\sec\theta$, $\quad dx = a\sec\theta\tan\theta\, d\theta$

Thus $\theta = \sec^{-1}(x/a)$. If $x/a \ge 1$, then $0 \le \theta < \pi/2$; if $x/a \le -1$, then $\pi/2 < \theta \le \pi$.

We restrict our work to where $x \ge a$, so $x/a \ge 1$, and $0 \le \theta < \pi/2$. On this interval, $\tan\theta \ge 0$, so

$\sqrt{x^2-a^2} = a\tan\theta$



Be fluent with these diagrams.
Be able to construct them
as needed.

Example 7.4.2 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{\sqrt{5+x^2}}\,dx$.

Solution Using Key Idea 6.4.1(b), we recognize $a=\sqrt{5}$ and set $x=\sqrt{5}\tan\theta$. This makes $dx=\sqrt{5}\sec^2\theta\,d\theta$. We will use the fact that $\sqrt{5+x^2}=\sqrt{5+5\tan^2\theta}=\sqrt{5\sec^2\theta}=\sqrt{5}\sec\theta$. Substituting, we have:

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \int \frac{1}{\sqrt{5+5\tan^2\theta}}\sqrt{5}\sec^2\theta\,d\theta\\ &= \int \frac{\sqrt{5}\sec^2\theta}{\sqrt{5}\sec\theta}\,d\theta\\ &= \int \sec\theta\,d\theta\\ &= \ln\big|\sec\theta+\tan\theta\big|+C.\end{aligned}$$

While the integration steps are over, we are not yet done. The original problem was stated in terms of x, whereas our answer is given in terms of θ. We must convert back to x.

The reference triangle given in Key Idea 6.4.1(b) helps. With $x=\sqrt{5}\tan\theta$, we have

$$\tan\theta=\frac{x}{\sqrt{5}} \quad\text{and}\quad \sec\theta=\frac{\sqrt{x^2+5}}{\sqrt{5}}.$$

This gives

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \ln\big|\sec\theta+\tan\theta\big|+C\\ &= \ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C.\end{aligned}$$

We can leave this answer as is, or we can use a logarithmic identity to simplify it. Note:

$$\begin{aligned}\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C &= \ln\left|\frac{1}{\sqrt{5}}\left(\sqrt{x^2+5}+x\right)\right|+C\\ &= \ln\left|\frac{1}{\sqrt{5}}\right|+\ln\left|\sqrt{x^2+5}+x\right|+C\\ &= \ln\left|\sqrt{x^2+5}+x\right|+C,\end{aligned}$$

where the $\ln\left(1/\sqrt{5}\right)$ term is absorbed into the constant C. (In Section 6.6 we will learn another way of approaching this problem.)

Example 7.4.3 Using Trigonometric Substitution

Evaluate $\displaystyle\int \sqrt{4x^2-1}\,dx$.

Solution We start by rewriting the integrand so that it looks like $\sqrt{x^2-a^2}$ for some value of a:

$$\begin{aligned}\sqrt{4x^2-1} &= \sqrt{4\left(x^2-\frac{1}{4}\right)}\\ &= 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}.\end{aligned}$$

So we have $a=1/2$, and following Key Idea 7.4.1(c), we set $x = \frac{1}{2}\sec\theta$, and hence $dx=\frac{1}{2}\sec\theta\tan\theta\,d\theta$. We now rewrite the integral with these substitutions:

$$\begin{aligned}\int\sqrt{4x^2-1}\,dx &= \int 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}\,dx\\ &= \int 2\sqrt{\frac{1}{4}\sec^2\theta-\frac{1}{4}}\left(\frac{1}{2}\sec\theta\tan\theta\right)\,d\theta\\ &= \int\sqrt{\frac{1}{4}(\sec^2\theta-1)}\Big(\sec\theta\tan\theta\Big)\,d\theta\\ &= \int\sqrt{\frac{1}{4}\tan^2\theta}\Big(\sec\theta\tan\theta\Big)\,d\theta\\ &= \int\frac{1}{2}\tan^2\theta\sec\theta\,d\theta\\ &= \frac{1}{2}\int\Big(\sec^2\theta-1\Big)\sec\theta\,d\theta\\ &= \frac{1}{2}\int\big(\sec^3\theta-\sec\theta\big)\,d\theta.\end{aligned}$$

We integrated $\sec^3\theta$ in Example 7.3.6, finding its antiderivatives to be

$$\int\sec^3\theta\,d\theta=\frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)+C.$$

Thus

$$\begin{aligned}\int\sqrt{4x^2-1}\,dx &= \frac{1}{2}\int\big(\sec^3\theta-\sec\theta\big)\,d\theta\\ &= \frac{1}{2}\left(\frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)-\ln|\sec\theta+\tan\theta|\right)+C\\ &= \frac{1}{4}\left(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right)+C.\end{aligned}$$

We are not yet done. Our original integral is given in terms of x, whereas our final answer, as given, is in terms of θ. We need to rewrite our answer in terms of x. With $a = 1/2$, and $x = \frac{1}{2}\sec\theta$, the reference triangle in Key Idea 6.4.1(c) shows that

$$\tan\theta = \sqrt{x^2 - 1/4}\Big/(1/2) = 2\sqrt{x^2 - 1/4} \quad \text{and} \quad \sec\theta = 2x.$$

Thus

$$\begin{aligned}\frac{1}{4}\Big(\sec\theta\tan\theta - \ln\big|\sec\theta + \tan\theta\big|\Big) + C &= \frac{1}{4}\Big(2x\cdot 2\sqrt{x^2 - 1/4} - \ln\big|2x + 2\sqrt{x^2 - 1/4}\big|\Big) + C \\ &= \frac{1}{4}\Big(4x\sqrt{x^2 - 1/4} - \ln\big|2x + 2\sqrt{x^2 - 1/4}\big|\Big) + C.\end{aligned}$$

The final answer is given in the last line above, repeated here:

$$\int \sqrt{4x^2 - 1}\,dx = \frac{1}{4}\Big(4x\sqrt{x^2 - 1/4} - \ln\big|2x + 2\sqrt{x^2 - 1/4}\big|\Big) + C.$$

Example 7.4.4 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{\sqrt{4 - x^2}}{x^2}\,dx.$

Solution We use Key Idea 7.4.1(a) with $a = 2$, $x = 2\sin\theta$, $dx = 2\cos\theta$ and hence $\sqrt{4 - x^2} = 2\cos\theta$. This gives

$$\begin{aligned}\int \frac{\sqrt{4 - x^2}}{x^2}\,dx &= \int \frac{2\cos\theta}{4\sin^2\theta}(2\cos\theta)\,d\theta \\ &= \int \cot^2\theta\,d\theta \\ &= \int (\csc^2\theta - 1)\,d\theta \\ &= -\cot\theta - \theta + C.\end{aligned}$$

We need to rewrite our answer in terms of x. Using the reference triangle found in Key Idea 6.4.1(a), we have $\cot\theta = \sqrt{4 - x^2}/x$ and $\theta = \sin^{-1}(x/2)$. Thus

$$\int \frac{\sqrt{4 - x^2}}{x^2}\,dx = -\frac{\sqrt{4 - x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C.$$

Trigonometric Substitution can be applied in many situations, even those not of the form $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{x^2 + a^2}$. In the following example, we apply it to an integral we already know how to handle.

Example 7.4.5 Using Trigonometric Substitution

Evaluate $\dfrac{1}{x^2+1}\,dx$.

Solution We know the answer already as $\tan^{-1} x + C$. We apply Trigonometric Substitution here to show that we get the same answer without inherently relying on knowledge of the derivative of the arctangent function.

Using Key Idea 7.4.1(b), let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$ and note that $x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$. Thus

$$\begin{aligned}
\int \frac{1}{x^2+1}\,dx &= \int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta\\
&= \int 1\,d\theta\\
&= \theta + C.
\end{aligned}$$

Since $x = \tan\theta$, $\theta = \tan^{-1} x$, and we conclude that $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$.

The next example is similar to the previous one in that it does not involve a square–root. It shows how several techniques and identities can be combined to obtain a solution.

Example 7.4.6 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{(x^2+6x+10)^2}\,dx$.

Solution We start by completing the square, then make the substitution $u = x + 3$, followed by the trigonometric substitution of $u = \tan\theta$:

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \int \frac{1}{\big((x+3)^2+1\big)^2}\,dx = \int \frac{1}{(u^2+1)^2}\,du.$$

Now make the substitution $u = \tan\theta$, $du = \sec^2\theta\,d\theta$:

$$\begin{aligned}
&= \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta\\
&= \int \frac{1}{(\sec^2\theta)^2}\sec^2\theta\,d\theta\\
&= \int \cos^2\theta\,d\theta.
\end{aligned}$$

Applying a power reducing formula, we have

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\, d\theta$$
$$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C. \quad (6.2)$$

We need to return to the variable x. As $u = \tan\theta$, $\theta = \tan^{-1} u$. Using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and using the reference triangle found in Key Idea 6.4.1(b), we have

$$\frac{1}{4}\sin(2\theta) = \frac{1}{2}\frac{u}{\sqrt{u^2+1}} \cdot \frac{1}{\sqrt{u^2+1}} = \frac{1}{2}\frac{u}{u^2+1}.$$

Finally, we return to x with the substitution $u = x+3$. We start with the expression in Equation (6.2):

$$\begin{aligned}\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C &= \frac{1}{2}\tan^{-1} u + \frac{1}{2}\frac{u}{u^2+1} + C \\ &= \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.\end{aligned}$$

Stating our final result in one line,

$$\int \frac{1}{(x^2+6x+10)^2}\, dx = \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.$$

Our last example returns us to definite integrals, as seen in our first example. Given a definite integral that can be evaluated using Trigonometric Substitution, we could first evaluate the corresponding indefinite integral (by changing from an integral in terms of x to one in terms of θ, then converting back to x) and then evaluate using the original bounds. It is much more straightforward, though, to change the bounds as we substitute.

Example 7.4.7 Definite integration and Trigonometric Substitution

Evaluate $\displaystyle\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\, dx$.

SOLUTION Using Key Idea 7.4.1(b), we set $x = 5\tan\theta$, $dx = 5\sec^2\theta\, d\theta$, and note that $\sqrt{x^2+25} = 5\sec\theta$. As we substitute, we can also change the bounds. The lower bound of the original integral is $x = 0$. As $x = 5\tan\theta$, we solve for θ and find $\theta = \tan^{-1}(x/5)$. Thus the new lower bound is $\theta = \tan^{-1}(0) = 0$. The original upper bound is $x = 5$, thus the new upper bound is $\theta = \tan^{-1}(5/5) = \pi/4$.

Thus we have

$$\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx = \int_0^{\pi/4} \frac{25\tan^2\theta}{5\sec\theta} 5\sec^2\theta\,d\theta$$
$$= 25\int_0^{\pi/4} \tan^2\theta\sec\theta\,d\theta.$$

We encountered this indefinite integral in Example 6.4.3 where we found

$$\int \tan^2\theta\sec\theta\,d\theta = \frac{1}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big).$$

So

$$25\int_0^{\pi/4} \tan^2\theta\sec\theta\,d\theta = \frac{25}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big)\Big|_0^{\pi/4}$$
$$= \frac{25}{2}\big(\sqrt{2} - \ln(\sqrt{2}+1)\big)$$
$$\approx 6.661.$$

The following equalities are very useful when evaluating integrals using Trigonometric Substitution.

Key Idea 7.4.2 Useful Equalities with Trigonometric Substitution

1. $\sin(2\theta) = 2\sin\theta\cos\theta$
2. $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
3. $\displaystyle\int \sec^3\theta\,d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta + \ln\big|\sec\theta+\tan\theta\big|\Big) + C$
4. $\displaystyle\int \cos^2\theta\,d\theta = \int \frac{1}{2}\big(1+\cos(2\theta)\big)\,d\theta = \frac{1}{2}\big(\theta + \sin\theta\cos\theta\big) + C.$

The next section introduces Partial Fraction Decomposition, which is an algebraic technique that turns “complicated” fractions into sums of “simpler” fractions, making integration easier.

If you are fluent with the trig substitution method, consider yourself a calculus master.

Exercises 7.4

Terms and Concepts

1. Trigonometric Substitution works on the same principles as Integration by Substitution, though it can feel "________".

2. If one uses Trigonometric Substitution on an integrand containing $\sqrt{25-x^2}$, then one should set $x =$________.

3. Consider the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$.
   (a) What identity is obtained when both sides are divided by $\cos^2\theta$?
   (b) Use the new identity to simplify $9\tan^2\theta + 9$.

4. Why does Key Idea 7.4.1(a) state that $\sqrt{a^2-x^2} = a\cos\theta$, and not $a|\cos\theta|$?

Problems

In Exercises 5 – 16, apply Trigonometric Substitution to evaluate the indefinite integrals.

5. $\displaystyle\int \sqrt{x^2+1}\,dx$

6. $\displaystyle\int \sqrt{x^2+4}\,dx$

7. $\displaystyle\int \sqrt{1-x^2}\,dx$

8. $\displaystyle\int \sqrt{9-x^2}\,dx$

9. $\displaystyle\int \sqrt{x^2-1}\,dx$

10. $\displaystyle\int \sqrt{x^2-16}\,dx$

11. $\displaystyle\int \sqrt{4x^2+1}\,dx$

12. $\displaystyle\int \sqrt{1-9x^2}\,dx$

13. $\displaystyle\int \sqrt{16x^2-1}\,dx$

14. $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\,dx$

15. $\displaystyle\int \frac{3}{\sqrt{7-x^2}}\,dx$

16. $\displaystyle\int \frac{5}{\sqrt{x^2-8}}\,dx$

In Exercises 17 – 26, evaluate the indefinite integrals. Some may be evaluated without Trigonometric Substitution.

17. $\displaystyle\int \frac{\sqrt{x^2-11}}{x}\,dx$

18. $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

19. $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\,dx$

20. $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

21. $\displaystyle\int \frac{x}{(x^2+9)^{3/2}}\,dx$

22. $\displaystyle\int \frac{5x^2}{\sqrt{x^2-10}}\,dx$

23. $\displaystyle\int \frac{1}{(x^2+4x+13)^2}\,dx$

24. $\displaystyle\int x^2(1-x^2)^{-3/2}\,dx$

25. $\displaystyle\int \frac{\sqrt{5-x^2}}{7x^2}\,dx$

26. $\displaystyle\int \frac{x^2}{\sqrt{x^2+3}}\,dx$

In Exercises 27 – 32, evaluate the definite integrals by making the proper trigonometric substitution *and* changing the bounds of integration. (Note: each of the corresponding indefinite integrals has appeared previously in this Exercise set.)

27. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$

28. $\displaystyle\int_{4}^{8} \sqrt{x^2-16}\,dx$

29. $\displaystyle\int_{0}^{2} \sqrt{x^2+4}\,dx$

30. $\displaystyle\int_{-1}^{1} \frac{1}{(x^2+1)^2}\,dx$

31. $\displaystyle\int_{-1}^{1} \sqrt{9-x^2}\,dx$

32. $\displaystyle\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx$

Solutions 7.4

1. backwards
2. $5\sin\theta$
3. (a) $\tan^2\theta+1=\sec^2\theta$

   (b) $9\sec^2\theta$.
4. Because we are considering $a>0$ and $x=a\sin\theta$, which means $\theta=\sin^{-1}(x/a)$. The arcsine function has a domain of $-\pi/2\le\theta\le\pi/2$; on this domain, $\cos\theta\ge 0$, so $a\cos\theta$ is always non-negative, allowing us to drop the absolute value signs.
5. $\frac{1}{2}\left(x\sqrt{x^2+1}+\ln|\sqrt{x^2+1}+x|\right)+C$
6. $2\left(\frac{x}{4}\sqrt{x^2+4}+\ln|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}|\right)+C$
7. $\frac{1}{2}\left(\sin^{-1}x+x\sqrt{1-x^2}\right)+C$
8. $\frac{1}{2}\left(9\sin^{-1}(x/3)+x\sqrt{9-x^2}\right)+C$
9. $\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln|x+\sqrt{x^2-1}|+C$
10. $\frac{1}{2}x\sqrt{x^2-16}-8\ln\left|\frac{x}{4}+\frac{\sqrt{x^2-16}}{4}\right|+C$
11. $x\sqrt{x^2+1/4}+\frac{1}{4}\ln|2\sqrt{x^2+1/4}+2x|+C=$ $\frac{1}{2}x\sqrt{4x^2+1}+\frac{1}{4}\ln|\sqrt{4x^2+1}+2x|+C$
12. $\frac{1}{6}\sin^{-1}(3x)+\frac{3}{2}\sqrt{1/9-x^2}+C=\frac{1}{6}\sin^{-1}(3x)+\frac{1}{2}\sqrt{1-9x^2}+C$
13. $4\left(\frac{1}{2}x\sqrt{x^2-1/16}-\frac{1}{32}\ln|4x+4\sqrt{x^2-1/16}|\right)+C=$ $\frac{1}{2}x\sqrt{16x^2-1}-\frac{1}{8}\ln|4x+\sqrt{16x^2-1}|+C$
14. $8\ln\left|\frac{\sqrt{x^2+2}}{\sqrt{2}}+\frac{x}{\sqrt{2}}\right|+C$; with Section 6.6, we can state the answer as $8\sinh^{-1}(x/\sqrt{2})+C$.
15. $3\sin^{-1}\left(\frac{x}{\sqrt{7}}\right)+C$ (Trig. Subst. is not needed)
16. $5\ln\left|\frac{x}{\sqrt{8}}+\frac{\sqrt{x^2-8}}{\sqrt{8}}\right|+C$
17. $\sqrt{x^2-11}-\sqrt{11}\sec^{-1}(x/\sqrt{11})+C$
18. $\frac{1}{2}\left(\tan^{-1}x+\frac{x}{x^2+1}\right)+C$
19. $\sqrt{x^2-3}+C$ (Trig. Subst. is not needed)
20. $\frac{1}{8}\sin^{-1}x-\frac{1}{8}x\sqrt{1-x^2}(1-2x^2)+C$
21. $-\frac{1}{\sqrt{x^2+9}}+C$ (Trig. Subst. is not needed)
22. $\frac{5}{2}x\sqrt{x^2-10}+25\ln\left|\frac{x}{\sqrt{10}}+\frac{\sqrt{x^2-10}}{\sqrt{10}}\right|+C$
23. $\frac{1}{18}\frac{x+2}{x^2+4x+13}+\frac{1}{54}\tan^{-1}\left(\frac{x+2}{2}\right)+C$
24. $\frac{x}{\sqrt{1-x^2}}-\sin^{-1}x+C$
25. $\frac{1}{7}\left(-\frac{\sqrt{5-x^2}}{x}-\sin^{-1}(x/\sqrt{5})\right)+C$
26. $\frac{1}{2}x\sqrt{x^2+3}-\frac{3}{2}\ln\left|\frac{\sqrt{x^2+3}}{\sqrt{3}}+\frac{x}{\sqrt{3}}\right|+C$
27. $\pi/2$
28. $16\sqrt{3}-8\ln(2+\sqrt{3})$
29. $2\sqrt{2}+2\ln(1+\sqrt{2})$
30. $\pi/4+1/2)$
31. $9\sin^{-1}(1/3)+\sqrt{8}$ Note: the new lower bound is $\theta=\sin^{-1}(-1/3)$ and the new upper bound is $\theta=\sin^{-1}(1/3)$. The final answer comes with recognizing that $\sin^{-1}(-1/3)=-\sin^{-1}(1/3)$ and that $\cos\left(\sin^{-1}(1/3)\right)=\cos\left(\sin^{-1}(-1/3)\right)=\sqrt{8}/3$.
32. $\pi/8$

7.5 Partial Fraction Decomposition

Understand this material and be able to work simple examples. It is important to understand that now, in theory, you can integrate about any rational function. Big people use a CAS for these problems.

In this section we investigate the antiderivatives of rational functions. Recall that rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Such functions arise in many contexts, one of which is the solving of certain fundamental differential equations.

We begin with an example that demonstrates the motivation behind this section. Consider the integral $\displaystyle\int \frac{1}{x^2-1}\,dx$. We do not have a simple formula for this (if the denominator were x^2+1, we would recognize the antiderivative as being the arctangent function). It can be solved using Trigonometric Substitution, but note how the integral is easy to evaluate once we realize:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

This section teaches how to *decompose*

$$\frac{1}{x^2-1} \quad \text{into} \quad \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

We start with a rational function $f(x) = \frac{p(x)}{q(x)}$, where p and q do not have any common factors and the degree of p is less than the degree of q. It can be shown that any polynomial, and hence q, can be factored into a product of linear and irreducible quadratic terms. The following Key Idea states how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

Note If the degree of the numerator is greater than or equal to that of the denominator, divide.

Example

$$\frac{x^4}{x^2+1}$$

$$= \left(x^3 - 1 + \frac{1}{x^2+1}\right) \qquad \text{polynomial division}$$

So

$$\int \frac{x^4}{x^2+1}\,dx$$

$$= \int \left(x^3 - 1 + \frac{1}{x^2+1}\right) dx$$

$$= \frac{x^4}{4} - x + \arctan x + C$$

Key Idea 7.5.1 Partial Fraction Decomposition

Let $\dfrac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q.

1. **Linear Terms:** Let $(x-a)$ divide $q(x)$, where $(x-a)^n$ is the highest power of $(x-a)$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum
$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_n}{(x-a)^n}.$$
2. **Quadratic Terms:** Let x^2+bx+c divide $q(x)$, where $(x^2+bx+c)^n$ is the highest power of x^2+bx+c that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum
$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

To find the coefficients A_i, B_i and C_i:

1. Multiply all fractions by $q(x)$, clearing the denominators. Collect like terms.
2. Equate the resulting coefficients of the powers of x and solve the resulting system of linear equations.

Example $\int \frac{x^4}{x^2+1}\,dx$

By polynomial division $\quad \frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}$

$$= \int \left(x^2 - 1 + \frac{1}{x^2+1}\right) dx$$

$$= \frac{x^3}{3} - x + \arctan x + C$$

The following examples will demonstrate how to put this Key Idea into prac-tice. Example 7.5.1 stresses the decomposition aspect of the Key Idea.

Example 7.5.1 Decomposing into partial fractions

Decompose $f(x)=\dfrac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}$ without solving for the resulting coefficients.

Solution The denominator is already factored, as both x^2+x+2 and x^2+x+7 cannot be factored further. We need to decompose $f(x)$ properly. Since $(x+5)$ is a linear term that divides the denominator, there will be a

$$\frac{A}{x+5}$$

term in the decomposition.

As $(x-2)^3$ divides the denominator, we will have the following terms in the decomposition:

$$\frac{B}{x-2},\quad \frac{C}{(x-2)^2}\quad \text{and}\quad \frac{D}{(x-2)^3}.$$

The x^2+x+2 term in the denominator results in a $\dfrac{Ex+F}{x^2+x+2}$ term.

Finally, the $(x^2+x+7)^2$ term results in the terms

$$\frac{Gx+H}{x^2+x+7}\quad \text{and}\quad \frac{Ix+J}{(x^2+x+7)^2}.$$

All together, we have

$$\begin{aligned}\frac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2} &= \frac{A}{x+5}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}+\\ &\quad \frac{Ex+F}{x^2+x+2}+\frac{Gx+H}{x^2+x+7}+\frac{Ix+J}{(x^2+x+7)^2}\end{aligned}$$

Solving for the coefficients $A, B \ldots J$ would be a bit tedious but not "hard."

Example 7.5.2 Decomposing into partial fractions

Perform the partial fraction decomposition of $\dfrac{1}{x^2-1}$.

Solution The denominator factors into two linear terms: $x^2-1 = (x-1)(x+1)$. Thus

$$\frac{1}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}.$$

To solve for A and B, first multiply through by $x^2-1=(x-1)(x+1)$:

$$\begin{aligned}1 &= \frac{A(x-1)(x+1)}{x-1}+\frac{B(x-1)(x+1)}{x+1}\\ &= A(x+1)+B(x-1)\\ &= Ax+A+Bx-B\end{aligned}$$

Now collect like terms.

$$= (A+B)x+(A-B).$$

The next step is key. Note the equality we have:

$$1=(A+B)x+(A-B).$$

For clarity's sake, rewrite the left hand side as

$$0x + 1 = (A + B)x + (A - B).$$

On the left, the coefficient of the x term is 0; on the right, it is $(A + B)$. Since both sides are equal, we must have that $0 = A + B$.

Likewise, on the left, we have a constant term of 1; on the right, the constant term is $(A - B)$. Therefore we have $1 = A - B$.

We have two linear equations with two unknowns. This one is easy to solve by hand, leading to

$$\begin{array}{c} A + B = 0 \\ A - B = 1 \end{array} \Rightarrow \begin{array}{c} A = 1/2 \\ B = -1/2 \end{array}.$$

Thus

$$\frac{1}{x^2 - 1} = \frac{1/2}{x - 1} - \frac{1/2}{x + 1}.$$

Example 7.5.3 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\, dx$.

Note: Equation7.3 offers a direct route to finding the values of A, B and C. Since the equation holds for all values of x, it holds in particular when $x = 1$. However, when $x = 1$, the right hand side simplifies to $A(1 + 2)^2 = 9A$. Since the left hand side is still 1, we have $1 = 9A$. Hence $A = 1/9$. Likewise, the equality holds when $x = -2$; this leads to the equation $1 = -3C$. Thus $C = -1/3$.
Knowing A and C, we can find the value of B by choosing yet another value of x, such as $x = 0$, and solving for B.

Solution We decompose the integrand as follows, as described by Key Idea 7.5.1:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

To solve for A, B and C, we multiply both sides by $(x-1)(x+2)^2$ and collect like terms:

$$\begin{aligned} 1 &= A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad (7.3) \\ &= Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C \\ &= (A+B)x^2 + (4A + B + C)x + (4A - 2B - C) \end{aligned}$$

We have

$$0x^2 + 0x + 1 = (A+B)x^2 + (4A + B + C)x + (4A - 2B - C)$$

leading to the equations

$$A + B = 0, \quad 4A + B + C = 0 \quad \text{and} \quad 4A - 2B - C = 1.$$

These three equations of three unknowns lead to a unique solution:

$$A = 1/9, \quad B = -1/9 \quad \text{and} \quad C = -1/3.$$

Thus

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{1/9}{x-1}\,dx + \int \frac{-1/9}{x+2}\,dx + \int \frac{-1/3}{(x+2)^2}\,dx.$$

Each can be integrated with a simple substitution with $u = x-1$ or $u = x+2$ (or by directly applying Key Idea 6.1.1 as the denominators are linear functions). The end result is

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C.$$

Example 7.5.4 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{x^3}{(x-5)(x+3)}\,dx.$

Solution Key Idea 7.5.1 presumes that the degree of the numerator is less than the degree of the denominator. Since this is not the case here, we begin by using polynomial division to reduce the degree of the numerator. We omit the steps, but encourage the reader to verify that

$$\frac{x^3}{(x-5)(x+3)} = x + 2 + \frac{19x+30}{(x-5)(x+3)}.$$

Note: The values of A and B can be quickly found using the technique described in the margin of Example 7.5.3.

Using Key Idea 6.5.1, we can rewrite the new rational function as:

$$\frac{19x+30}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

for appropriate values of A and B. Clearing denominators, we have

$$\begin{aligned} 19x + 30 &= A(x+3) + B(x-5) \\ &= (A+B)x + (3A-5B). \end{aligned}$$

This implies that:

$$\begin{aligned} 19 &= A + B \\ 30 &= 3A - 5B. \end{aligned}$$

Solving this system of linear equations gives

$$\begin{aligned} 125/8 &= A \\ 27/8 &= B. \end{aligned}$$

We can now integrate.

$$\begin{aligned}\int \frac{x^3}{(x-5)(x+3)}\,dx &= \int \left(x+2+\frac{125/8}{x-5}+\frac{27/8}{x+3}\right)\,dx \\ &= \frac{x^2}{2}+2x+\frac{125}{8}\ln|x-5|+\frac{27}{8}\ln|x+3|+C.\end{aligned}$$

Example 7.5.5 Integrating using partial fractions

Use partial fraction decomposition to evaluate $\displaystyle\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx$.

Solution The degree of the numerator is less than the degree of the denominator so we begin by applying Key Idea 6.5.1. We have:

$$\frac{7x^2+31x+54}{(x+1)(x^2+6x+11)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+6x+11}.$$

Now clear the denominators.

$$\begin{aligned}7x^2+31x+54 &= A(x^2+6x+11)+(Bx+C)(x+1)\\ &= (A+B)x^2+(6A+B+C)x+(11A+C).\end{aligned}$$

This implies that:

$$\begin{aligned}7 &= A+B\\ 31 &= 6A+B+C\\ 54 &= 11A+C.\end{aligned}$$

Solving this system of linear equations gives the nice result of $A=5$, $B=2$ and $C=-1$. Thus

$$\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx = \int\left(\frac{5}{x+1}+\frac{2x-1}{x^2+6x+11}\right)\,dx.$$

The first term of this new integrand is easy to evaluate; it leads to a $5\ln|x+1|$ term. The second term is not hard, but takes several steps and uses substitution techniques.

The integrand $\dfrac{2x-1}{x^2+6x+11}$ has a quadratic in the denominator and a linear term in the numerator. This leads us to try substitution. Let $u=x^2+6x+11$, so $du=(2x+6)\,dx$. The numerator is $2x-1$, not $2x+6$, but we can get a $2x+6$

term in the numerator by adding 0 in the form of "$7 - 7$."

$$\begin{aligned}\frac{2x-1}{x^2+6x+11} &= \frac{2x-1+7-7}{x^2+6x+11}\\ &= \frac{2x+6}{x^2+6x+11} - \frac{7}{x^2+6x+11}.\end{aligned}$$

We can now integrate the first term with substitution, leading to a $\ln|x^2+6x+11|$ term. The final term can be integrated using arctangent. First, complete the square in the denominator:

$$\frac{7}{x^2+6x+11} = \frac{7}{(x+3)^2+2}.$$

An antiderivative of the latter term can be found using Theorem 6.1.3 and substitution:

$$\int \frac{7}{x^2+6x+11}\,dx = \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.$$

Let's start at the beginning and put all of the steps together.

$$\begin{aligned}\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx &= \int \left(\frac{5}{x+1} + \frac{2x-1}{x^2+6x+11}\right)dx\\ &= \int \frac{5}{x+1}\,dx + \int \frac{2x+6}{x^2+6x+11}\,dx - \int \frac{7}{x^2+6x+11}\,dx\\ &= 5\ln|x+1| + \ln|x^2+6x+11| - \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.\end{aligned}$$

As with many other problems in calculus, it is important to remember that one is not expected to "see" the final answer immediately after seeing the problem. Rather, given the initial problem, we break it down into smaller problems that are easier to solve. The final answer is a combination of the answers of the smaller problems.

Partial Fraction Decomposition is an important tool when dealing with rational functions. Note that at its heart, it is a technique of algebra, not calculus, as we are rewriting a fraction in a new form. Regardless, it is very useful in the realm of calculus as it lets us evaluate a certain set of "complicated" integrals.

Exercises 7.5

Terms and Concepts

1. Fill in the blank: Partial Fraction Decomposition is a method of rewriting ________ functions.

2. T/F: It is sometimes necessary to use polynomial division before using Partial Fraction Decomposition.

3. Decompose $\dfrac{1}{x^2-3x}$ without solving for the coefficients, as done in Example 6.5.1.

4. Decompose $\dfrac{7-x}{x^2-9}$ without solving for the coefficients, as done in Example 6.5.1.

5. Decompose $\dfrac{x-3}{x^2-7}$ without solving for the coefficients, as done in Example 6.5.1.

6. Decompose $\dfrac{2x+5}{x^3+7x}$ without solving for the coefficients, as done in Example 7.5.1.

Problems

In Exercises 7 – 26, evaluate the indefinite integral.

7. $\displaystyle\int \frac{7x+7}{x^2+3x-10}\,dx$

8. $\displaystyle\int \frac{7x-2}{x^2+x}\,dx$

9. $\displaystyle\int \frac{-4}{3x^2-12}\,dx$

10. $\displaystyle\int \frac{6x+4}{3x^2+4x+1}\,dx$

11. $\displaystyle\int \frac{x+7}{(x+5)^2}\,dx$

12. $\displaystyle\int \frac{-3x-20}{(x+8)^2}\,dx$

13. $\displaystyle\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx$

14. $\displaystyle\int \frac{-12x^2-x+33}{(x-1)(x+3)(3-2x)}\,dx$

15. $\displaystyle\int \frac{94x^2-10x}{(7x+3)(5x-1)(3x-1)}\,dx$

16. $\displaystyle\int \frac{x^2+x+1}{x^2+x-2}\,dx$

17. $\displaystyle\int \frac{x^3}{x^2-x-20}\,dx$

18. $\displaystyle\int \frac{2x^2-4x+6}{x^2-2x+3}\,dx$

19. $\displaystyle\int \frac{1}{x^3+2x^2+3x}\,dx$

20. $\displaystyle\int \frac{x^2+x+5}{x^2+4x+10}\,dx$

Do the remaining problems using a CAS or Wolfram Alpha.

21. $\displaystyle\int \frac{12x^2+21x+3}{(x+1)(3x^2+5x-1)}\,dx$

22. $\displaystyle\int \frac{6x^2+8x-4}{(x-3)(x^2+6x+10)}\,dx$

23. $\displaystyle\int \frac{2x^2+x+1}{(x+1)(x^2+9)}\,dx$

24. $\displaystyle\int \frac{x^2-20x-69}{(x-7)(x^2+2x+17)}\,dx$

25. $\displaystyle\int \frac{9x^2-60x+33}{(x-9)(x^2-2x+11)}\,dx$

26. $\displaystyle\int \frac{6x^2+45x+121}{(x+2)(x^2+10x+27)}\,dx$

In Exercises 27 – 30, evaluate the definite integral.

27. $\displaystyle\int_1^2 \frac{8x+21}{(x+2)(x+3)}\,dx$

28. $\displaystyle\int_0^5 \frac{14x+6}{(3x+2)(x+4)}\,dx$

29. $\displaystyle\int_{-1}^1 \frac{x^2+5x-5}{(x-10)(x^2+4x+5)}\,dx$

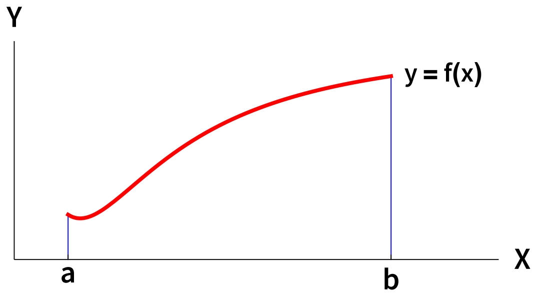
30. $\displaystyle\int_0^1 \frac{x}{(x+1)(x^2+2x+1)}\,dx$

Solutions 7.5

1. rational
2. T
3. $\frac{A}{x}+\frac{B}{x-3}$
4. $\frac{A}{x-3}+\frac{B}{x+3}$
5. $\frac{A}{x-\sqrt{7}}+\frac{B}{x+\sqrt{7}}$
6. $\frac{A}{x}+\frac{Bx+C}{x^2+7}$
7. $3\ln|x-2|+4\ln|x+5|+C$
8. $9\ln|x+1|-2\ln|x|+C$
9. $\frac{1}{3}(\ln|x+2|-\ln|x-2|)+C$
10. $\ln|x+1|+\ln|3x+1|+C$
11. $\ln|x+5|-\frac{2}{x+5}+C$
12. $-\frac{4}{x+8}-3\ln|x+8|+C$
13. $\frac{5}{x+1}+7\ln|x|+2\ln|x+1|+C$
14. $-\ln|2x-3|+5\ln|x-1|+2\ln|x+3|+C$
15. $-\frac{1}{5}\ln|5x-1|+\frac{2}{3}\ln|3x-1|+\frac{3}{7}\ln|7x+3|+C$
16. $x+\ln|x-1|-\ln|x+2|+C$
17. $\frac{x^2}{2}+x+\frac{125}{9}\ln|x-5|+\frac{64}{9}\ln|x+4|-\frac{35}{2}+C$
18. $2x+C$
19. $\frac{1}{6}\left(-\ln\left|x^2+2x+3\right|+2\ln|x|-\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right)+C$
20. $-\frac{3}{2}\ln\left|x^2+4x+10\right|+x+\frac{\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right)}{\sqrt{6}}+C$
21. $\ln\left|3x^2+5x-1\right|+2\ln|x+1|+C$
22. $2\ln|x-3|+2\ln|x^2+6x+10|-4\tan^{-1}(x+3)+C$
23. $\frac{9}{10}\ln\left|x^2+9\right|+\frac{1}{5}\ln|x+1|-\frac{4}{15}\tan^{-1}\left(\frac{x}{3}\right)+C$
24. $\frac{1}{2}\left(3\ln\left|x^2+2x+17\right|-4\ln|x-7|+\tan^{-1}\left(\frac{x+1}{4}\right)\right)+C$
25. $3\left(\ln\left|x^2-2x+11\right|+\ln|x-9|\right)+3\sqrt{\frac{2}{5}}\tan^{-1}\left(\frac{x-1}{\sqrt{10}}\right)+C$
26. $\frac{1}{2}\ln\left|x^2+10x+27\right|+5\ln|x+2|-6\sqrt{2}\tan^{-1}\left(\frac{x+5}{\sqrt{2}}\right)+C$
27. $\ln(2000/243)\doteq 2.108$
28. $5\ln(9/4)-\ln(17/2)\doteq 3.3413$
29. $-\pi/4+\tan^{-1}3-\ln(11/9)\doteq 0.263$
30. $1/8$

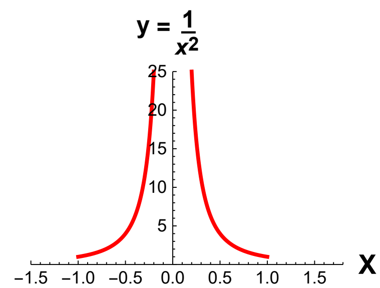
7.6 Improper Integrals

In the definition of definite integral, $\int_a^b f(x)\,dx$, it was assumed that the integrand was bounded (not infinite) on the interval of integration. Also it was assumed that the limits of integration a and b were real numbers, not the extended reals, $-\infty$ or $+\infty$.



In applications, these restrictions are unnecessary and undesirable. With one caution*, these new type of integrals can be evaluated in the usual way.

***Caution:** If $f(x)$ is infinite at an interior point x of the interval, $a < x < b$, break up the interval so that the infinity occurs at endpoints.



Example Evaluate $\int_{-1}^{1} \frac{dx}{x^2}$.

$$= -\frac{1}{x}\Big|_{-1}^{1}$$

$$= -\frac{1}{1} - \left(-\frac{1}{-1}\right)$$

$$= -2.$$

Wrong! The answer should be positive since $f(x) > 0$ on the interval $-1 \le x \le 1$.

Normally do not integrate *across* a point where the integrand is infinite.

Correct:

$$\int_{-1}^{1} \frac{dx}{x^2} \approx \int_{-1}^{0^-} \frac{dx}{x^2} + \int_{0^+}^{1} \frac{dx}{x^2}$$

$$= -\frac{1}{x}\Big|_{-1}^{0^-} + -\frac{1}{x}\Big|_{0^+}^{1}$$ hyperreal arithmetic

$$= \left(-\frac{1}{0^-} - -\frac{1}{-1}\right) + \left(-\frac{1}{1} - -\frac{1}{0^+}\right)$$

$$= (+\infty - 1) + (-1 + \infty)$$

$$= 2(+\infty) - 2$$

$$= +\infty.$$

Note: We allow $+\infty$ as an answer because in all applications this answer is meaningful. If this is a 'find the area under the curve' problem, it would cost infinite many $'s to buy the paint to coat it.

Type I Improper Integrals: infinite limits of integration
Type II Improper Integrals: infinite integrands
Mixed Type I, Type II Integrals

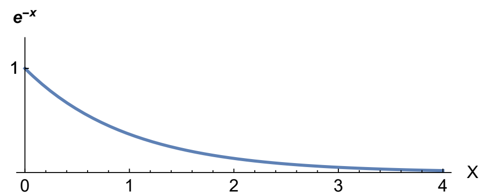
With hyperreal methods, these categories are not very relevant.

Possible Outcomes:

- **An extended real number**
- **Does not exist**

Note in your Apex readings that the author uses limit methods. The limit method is more prone toward making errors. Hyperreal methods also seem more natural; you work these just like you did for 'proper integrals'.

Example Type I

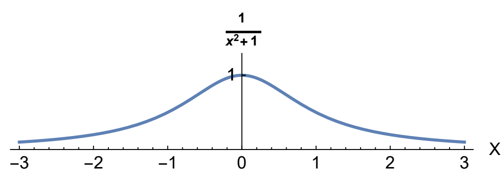


$$\int_0^{+\infty} e^{-x}\,dx = -e^{-x}\Big|_0^{+\infty}$$
$$= -e^{-\infty} - (-e^0)$$
$$= 0 + 1$$
$$= 1$$

Use the more natural hyperreal notation when doing improper integrals:

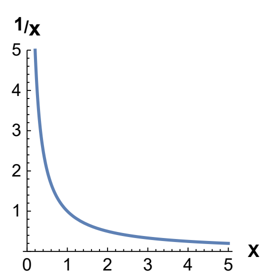
$0^+ = dx,\ \ dx > 0$
$0^- = -dx,\ \ dx > 0$
$a^+ = a + 0^+$, etc.

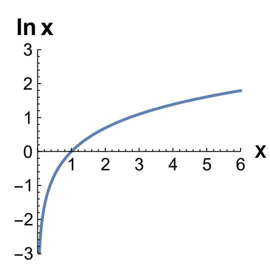
Example Type I



$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = \arctan x\Big|_{-\infty}^{+\infty}$$
$$= \arctan(+\infty) - \arctan(-\infty)$$
$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$
$$= \pi$$

Example Mixed Type





$$\int_0^{+\infty} \frac{dx}{x} = \ln x\Big|_{0^+}^{+\infty}$$
$$= \ln(+\infty) - \ln(0^+)$$
$$= +\infty - (-\infty)$$
$$= \{\infty + \infty\}$$
$$= +\infty$$

Exercise Show that $\int_{-\infty}^{+\infty} \frac{x\,dx}{x^2+1}$ does not exists because it is the indeterminate form $\{\infty - \infty\}$.

Answer: See the front cover or the next page

Improper Integration Readings

We begin this section by considering the following definite integrals:

- $\displaystyle\int_0^{100} \frac{1}{1+x^2}\,dx \doteq 1.5608,$
- $\displaystyle\int_0^{1000} \frac{1}{1+x^2}\,dx \doteq 1.5698,$
- $\displaystyle\int_0^{10,000} \frac{1}{1+x^2}\,dx \doteq 1.5707.$

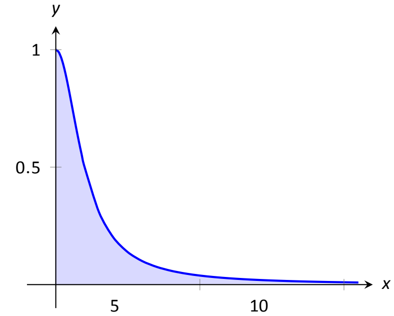


Figure 7.6.1: Graphing $f(x) = \frac{1}{1+x^2}$

Notice how the integrand is $1/(1+x^2)$ in each integral (which is sketched in Figure 7.8.1). As the upper bound gets larger, one would expect the "area under the curve" would also grow. While the definite integrals do increase in value as the upper bound grows, they are not increasing by much. In fact, consider:

$$\int_0^b \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^b = \tan^{-1} b - \tan^{-1} 0 = \tan^{-1} b.$$

As $b \to \infty$, $\tan^{-1} b \to \pi/2$. Therefore it seems that as the upper bound b grows, the value of the definite integral $\displaystyle\int_0^b \frac{1}{1+x^2}\,dx$ approaches $\pi/2 \doteq 1.5708$. This should strike the reader as being a bit amazing: even though the curve extends "to infinity," it has a finite amount of area underneath it.

When we defined the definite integral $\displaystyle\int_a^b f(x)\,dx$, we made two stipulations:

1. The interval over which we integrated, $[a,b]$, was a finite interval, and
2. The function $f(x)$ was continuous on $[a,b]$ (ensuring that the range of f was finite).

In this section we consider integrals where one or both of the above conditions do not hold. Such integrals are called **improper integrals.**

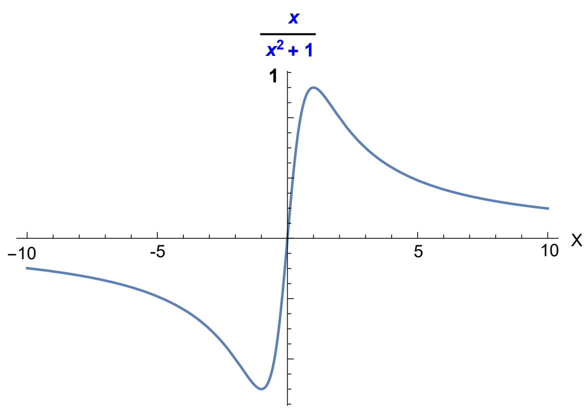
Note: when using hyperreal methods, the only time you should break up the interval of integration is when the the integrand has an infinite value in the interior of the interval of integration.

NOTE In a previous example we saw that

$$\int_{-\infty}^{+\infty} \frac{x\,dx}{1+x^2} = \{\infty - \infty\}$$ **, indeterminate. The integral does not exist.**

One might think that, by symmetry about the origin, the answer should be 0. But we always agree that this integral does not exist. In applications this agreement also often makes sense. Think about why this is true.

Nevertheless, in some advanced applications, there is a variation called the Cauchy Principal Value of the integral which in this case is 0. In this course we will not allow it.



Improper Integrals with Infinite Bounds

Definition 7.6.1 Improper Integrals with Infinite Bounds; Converge, Diverge

1. Let f be a continuous function on $[a,\infty)$. Define

$$\int_a^{\infty} f(x)\,dx \quad \text{to be} \quad \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. Let f be a continuous function on $(-\infty,b]$. Define

$$\int_{-\infty}^{b} f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. Let f be a continuous function on $(-\infty,\infty)$. Let c be any real number; define

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx.$$

An improper integral is said to **converge** if its corresponding limit exists; otherwise, it **diverges**. The improper integral in part 3 converges if and only if both of its limits exist.

Limit Talk.
Grain of salt material.

Example 7.6.1 Evaluating improper integrals
Evaluate the following improper integrals.

We prefer not to use the term '***improper***'.

1. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$
3. $\displaystyle\int_{-\infty}^{0} e^x\,dx$
4. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Solution

1. $$\int_1^{\infty}\frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left.\frac{-1}{x}\right|_1^b = \lim_{b\to\infty}\frac{-1}{b}+1 = 1.$$

A graph of the area defined by this integral is given in Figure 7.6.2.

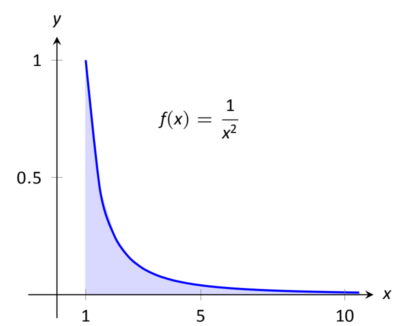
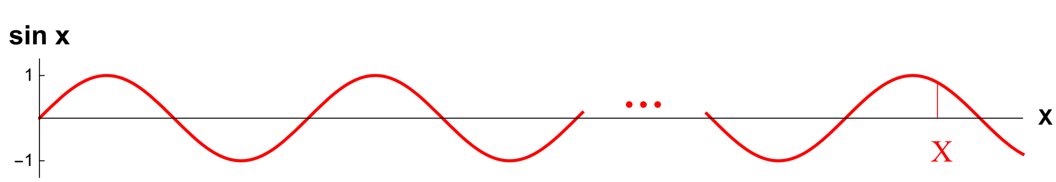


Figure 7.6.2: A graph of $f(x)=\frac{1}{x^2}$ in Example 7.6.1.

Example $\int_0^{+\infty}\sin x\,dx$ does not exist because the answer depends on the positive infinite number X.



2. $$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty} \ln|x|\Big|_1^b$$
$$= \lim_{b\to\infty} \ln(b)$$
$$= +\infty.$$

The limit does not exist, hence the improper integral $\int_1^\infty \frac{1}{x}\,dx$ diverges. Compare the graphs in Figures 7.6.2 and 7.6.3; notice how the graph of $f(x) = 1/x$ is noticeably larger. This difference is enough to cause the improper integral to diverge.

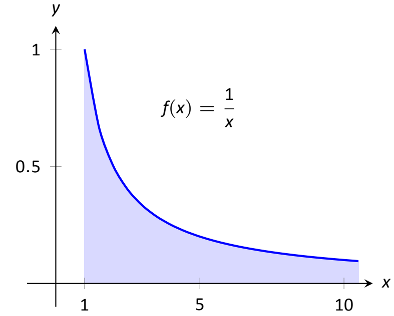


Figure 7.6.3: A graph of $f(x) = \frac{1}{x}$ in Example 7.6.1.

3. $$\int_{-\infty}^0 e^x\,dx = \lim_{a\to-\infty}\int_a^0 e^x\,dx$$
$$= \lim_{a\to-\infty} e^x\Big|_a^0$$
$$= \lim_{a\to-\infty} e^0 - e^a$$
$$= 1.$$

A graph of the area defined by this integral is given in Figure 7.6.4.

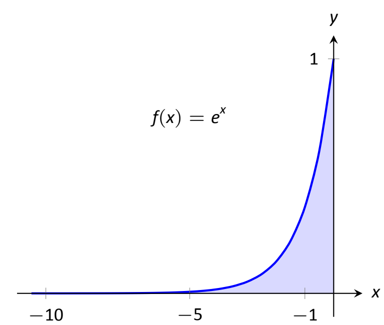


Figure 7.6.4: A graph of $f(x) = e^x$ in Ex-ample 7.6.1.

4. We will need to break this into two improper integrals and choose a value of c as in part 3 of Definition 7.6.1. Any value of c is fine; we choose $c = 0$.

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx$$
$$= \lim_{a\to-\infty} \tan^{-1}x\Big|_a^0 + \lim_{b\to\infty}\tan^{-1}x\Big|_0^b$$
$$= \lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}a\right) + \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right)$$
$$= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right).$$

Each limit exists, hence the original integral converges and has value:

$$= \pi.$$

A graph of the area defined by this integral is given in Figure 7.6.5.

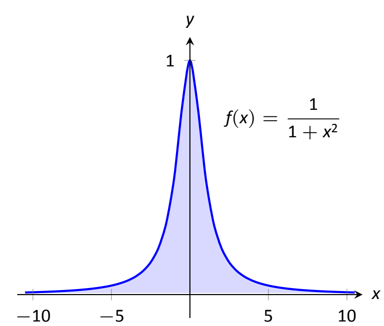
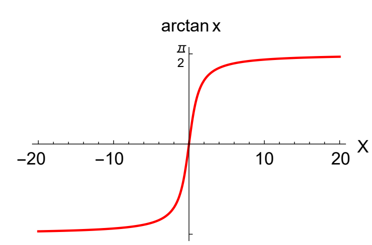


Figure 7.6.5: A graph of $f(x) = \frac{1}{1+x^2}$ in Example 7.6.1.

Hyperreally

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = \arctan x\,\Big|_{-\infty}^{+\infty} = \arctan(+\infty) - \arctan(-\infty) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$



The previous section introduced l'Hôpital's Rule, a method of evaluating limits that return indeterminate forms. It is not uncommon for the limits resulting from improper integrals to need this rule as demonstrated next.

Example 7.6.2 Improper integration and l'Hôpital's Rule **Be aware of this**

Evaluate the improper integral $\int_1^{\infty} \frac{\ln x}{x^2}\, dx$.

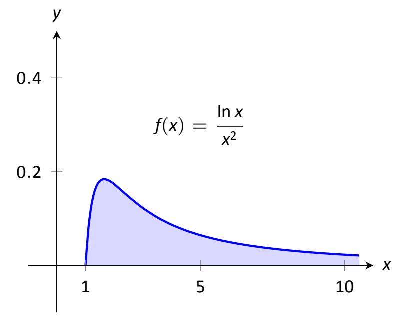


Figure 7.6.6: A graph of $f(x) = \frac{\ln x}{x^2}$ in Example 7.6.2.

Solution This integral will require the use of Integration by Parts. Let $u = \ln x$ and $dv = 1/x^2\, dx$. Then

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{x^2}\, dx &= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\, dx \\
&= \lim_{b\to\infty} \left(-\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\, dx \right) \\
&= \lim_{b\to\infty} \left(-\frac{\ln x}{x} - \frac{1}{x} \right)\Big|_1^b \\
&= \lim_{b\to\infty} \left(-\frac{\ln b}{b} - \frac{1}{b} - (-\ln 1 - 1) \right).
\end{aligned}$$

The $1/b$ and $\ln 1$ terms go to 0, leaving $\lim_{b\to\infty} -\frac{\ln b}{b} + 1$. We need to evaluate $\lim_{b\to\infty} \frac{\ln b}{b}$ with l'Hôpital's Rule. We have:

$$\begin{aligned}
\lim_{b\to\infty} \frac{\ln b}{b} &\overset{\text{by LHR}}{=} \lim_{b\to\infty} \frac{1/b}{1} \\
&= 0.
\end{aligned}$$

Thus the improper integral evaluates as:

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = 1.$$

Improper Integrals with Infinite Range

We have just considered definite integrals where the interval of integration was infinite. We now consider another type of improper integration, where the range of the integrand is infinite.

Definition 7.6.2 Improper Integration with Infinite Range

Let $f(x)$ be a continuous function on $[a,b]$ except at c, $a \le c \le b$, where $x = c$ is a vertical asymptote of f. Define

$$\int_a^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx.$$

Note: In Definition 7.6.2, c can be one of the endpoints (a or b). In that case, there is only one limit to consider as part of the definition.

Example 7.6.3 Improper integration of functions with infinite range

Evaluate the following improper integrals:

1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
2. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx.$

SOLUTION

1. A graph of $f(x) = 1/\sqrt{x}$ is given in Figure 7.6.7. Notice that f has a vertical asymptote at $x = 0$; in some sense, we are trying to compute the area of a region that has no "top." Could this have a finite value?

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1\\ &= \lim_{a\to 0^+} 2\left(\sqrt{1}-\sqrt{a}\right)\\ &= 2.\end{aligned}$$

It turns out that the region does have a finite area even though it has no upper bound (strange things can occur in mathematics when considering the infinite).

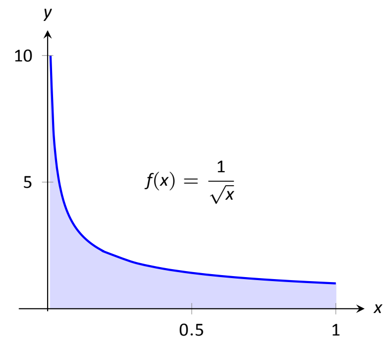


Figure 7.6.7: A graph of $f(x) = \frac{1}{\sqrt{x}}$ in Example 7.6.3.

2. The function $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$, as shown in Figure 7.6.8, so this integral is an improper integral. Let's eschew using limits for a moment and proceed without recognizing the improper nature of the integral. This leads to:

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\,dx &= -\frac{1}{x}\Big|_{-1}^1\\ &= -1-(1)\\ &= -2.\ (!)\end{aligned}$$

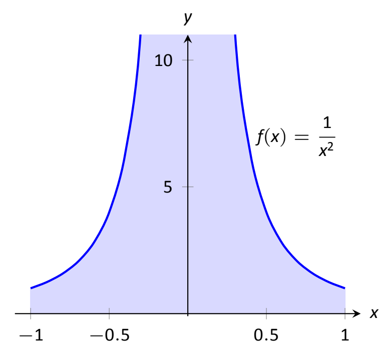


Figure 7.6.8: A graph of $f(x) = \frac{1}{x^2}$ in Example 7.6.3.

Clearly the area in question is above the x-axis, yet the area is supposedly negative! Why does our answer not match our intuition? To answer this, evaluate the integral using Definition 7.6.2.

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\,dx &= \lim_{t\to 0^-}\int_{-1}^t \frac{1}{x^2}\,dx + \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx\\ &= \lim_{t\to 0^-} -\frac{1}{x}\Big|_{-1}^t + \lim_{t\to 0^+} -\frac{1}{x}\Big|_t^1\\ &= \lim_{t\to 0^-} -\frac{1}{t} - 1 + \lim_{t\to 0^+} -1 + \frac{1}{t}\\ &\Rightarrow \left(\infty - 1\right) + \left(-1+\infty\right).\end{aligned}$$

Neither limit converges hence the original improper integral diverges. The nonsensical answer we obtained by ignoring the improper nature of the integral is just that: nonsensical*.

*Actually, the answer, $+\infty$, is completely sensical.

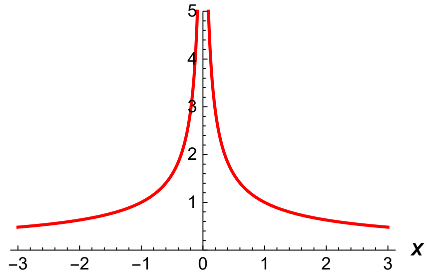
If this is an area problem, you would not be able to afford to buy enough paint to cover it.

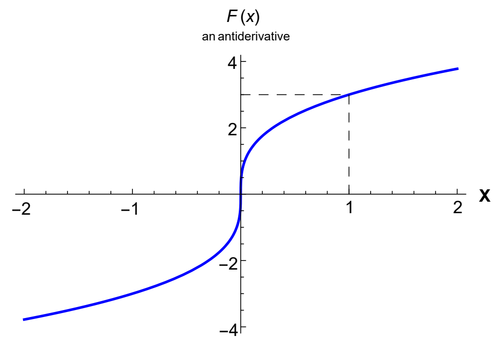
If this was a work problem, you could not afford to purchase enough energy to do the project.

It is good practice not to integrate across an infinite discontinuity. However, this can be done if the antiderivative is continuous at the discontinuity of the integrand.

Example

$$f(x) = \frac{1}{|x|^{2/3}}$$





NOTE The details of finding F(x) are somewhat complicated and are omitted.
Try it with a CAS or Wolfram Alpha.

Clearly, $\int_{-1}^{1} \frac{dx}{|x|^{2/3}}$ = F(1) - F(-1) = 3 - (-3) = 6

Understanding Convergence and Divergence

Be aware of this topic. Read carefully, but don't memorize.

Oftentimes, but not often, we are interested in knowing simply whether or not an improper integral converges, and not necessarily the value of a convergent integral. We provide here several tools that help determine the convergence or divergence of improper integrals without integrating.

Our first tool is to understand the behavior of functions of the form $\dfrac{1}{x^p}$.

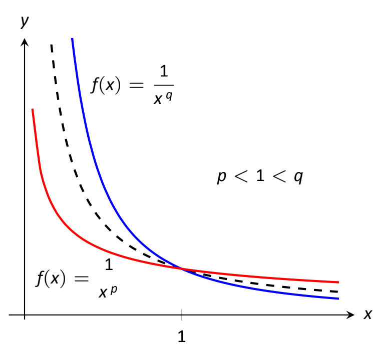


Figure 7.6.9: Plotting functions of the form $1/x^p$ in Example 7.6.4.

Example 6.8.4 Improper integration of $1/x^p$

Determine the values of p for which $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges.

SOLUTION We begin by integrating and then evaluating the limit.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx \\
&= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \qquad (\text{assume } p \neq 1) \\
&= \lim_{b\to\infty} \frac{1}{-p+1}x^{-p+1}\Big|_1^b \\
&= \lim_{b\to\infty} \frac{1}{1-p}\left(b^{1-p} - 1^{1-p}\right).
\end{aligned}$$

When does this limit converge – i.e., when is this limit *not* ∞? This limit converges precisely when the power of b is less than 0: when $1 - p < 0 \Rightarrow 1 < p$.

If $p \leqslant 1$, the integral is $+\infty$.
If $p > 1$, the integral is a real number.

Our analysis shows that if $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,dx$ converges. When $p < 1$ the improper integral diverges; we showed in Example 7.8.1 that when $p = 1$ the integral also diverges.

Figure 7.6.9 graphs $y = 1/x$ with a dashed line, along with graphs of $y = 1/x^p$, $p < 1$, and $y = 1/x^q$, $q > 1$. Somehow the dashed line forms a dividing line between convergence and divergence.

The result of Example 7.6.4 provides an important tool in determining the convergence of other integrals. A similar result is proved in the exercises about improper integrals of the form $\int_0^1 \frac{1}{x^p}\,dx$. These results are summarized in the following Key Idea.

Key Idea 7.6.1 Convergence of Improper Integrals $\int_1^\infty \frac{1}{x^p}\,dx$ and $\int_0^1 \frac{1}{x^p}\,dx$.

1. The improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p > 1$ and diverges when $p \leq 1$.
2. The improper integral $\int_0^1 \frac{1}{x^p}\,dx$ converges when $p < 1$ and diverges when $p \geq 1$.

We do not use the diverge/converge terminology.

A basic technique in determining convergence of improper integrals is to compare an integrand whose convergence is unknown to an integrand whose convergence is known. We often use integrands of the form $1/x^p$ to compare to as their convergence on certain intervals is known. This is described in the following theorem.

Note: We used the upper and lower bound of "1" in Key Idea 7.6.1 for convenience. It can be replaced by any a where $a > 0$.

Theorem 7.6.1 Direct Comparison Test for Improper Integrals

Let f and g be continuous on $[a, \infty)$ where $0 \leq f(x) \leq g(x)$.

1. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

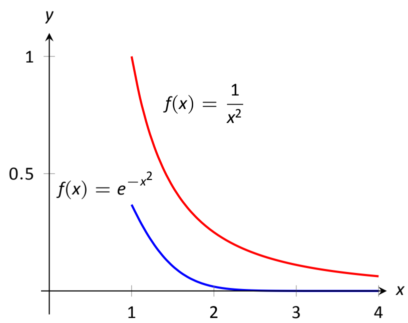


Figure 7.6.10: Graphs of $f(x) = e^{-x^2}$ and $f(x) = 1/x^2$ in Example 7.8.5.

Example 7.6.5 Determining convergence of improper integrals

Determine the convergence ofthe following improper integrals.

1. $\displaystyle\int_1^{\infty} e^{-x^2}\,dx$ 2. $\displaystyle\int_3^{\infty} \frac{1}{\sqrt{x^2 - x}}\,dx$

Solution

1. The function $f(x) = e^{-x^2}$ does not have an antiderivative expressible in terms of elementary functions, so we cannot integrate directly. It is comparable to $g(x) = 1/x^2$, and as demonstrated in Figure 7.6.10, $e^{-x^2} < 1/x^2$ on $[1, \infty)$. We know from Key Idea 7.6.1 $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ converges, hence $\displaystyle\int_1^{\infty} e^{-x^2}\,dx$ also converges.

2. Note that for large values of x, $\displaystyle\frac{1}{\sqrt{x^2 - x}} \doteq \frac{1}{\sqrt{x^2}} = \frac{1}{x}$. We know from Key Idea 7.6.1 and the subsequent note that to $\displaystyle\int_3^{\infty} \frac{1}{x}\,dx$ diverges, so we seek compare the original integrand to 1/x.

It is easy to see that when $x > 0$, we have $x = \sqrt{x^2} > \sqrt{x^2 - x}$. R_igle reciprocals reverses the inequality, giving

$$\frac{1}{x} < \frac{1}{\sqrt{x^2 - x}}.$$

Using Theorem 7.8.1, we conclude that since $\displaystyle\int_3^{\infty} \frac{1}{x}\,dx$ diverges, $\displaystyle\int_3^{\infty} \frac{1}{\sqrt{x^2 - x}}\,dx$ diverges as well. Figure 7.8.11 illustrates this.

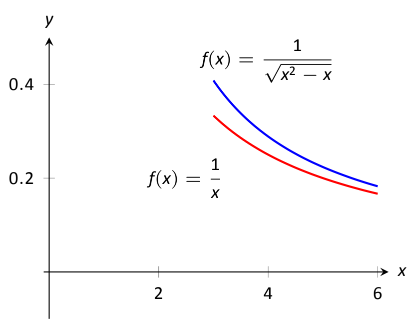


Figure 7.6.11: Graphs of $f(x) = 1/\sqrt{x^2 - x}$ and $f(x) = 1/x$ in Example 7.8.5.

Being able to compare "unknown" integrals to "known" integrals is very useful in determining convergence. However, some of our examples were a little "too nice." For instance, it was convenient that $\displaystyle\frac{1}{x} < \frac{1}{\sqrt{x^2 - x}}$, but what if the "$-x$" were replaced with a "$+2x + 5$"? That is, what can we say about the convergence of $\displaystyle\int_3^{\infty} \frac{1}{\sqrt{x^2 + 2x + 5}}\,dx$? We have $\displaystyle\frac{1}{x} > \frac{1}{\sqrt{x^2 + 2x + 5}}$, so we cannot use Theorem 7.8.1.

In cases like this (and many more) it is useful to employ the following theorem.

Theorem 7.6.2 Limit Comparison Test for Improper Integrals

Let f and g be continuous functions on $[a, \infty)$ where $f(x) > 0$ and $g(x) > 0$ for all x. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

Example 7.6.6 Determining convergence of improper integrals

Determine the convergence of $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

Solution As x gets large, the denominator of the integrand will begin to behave much like $y = x$. So we compare $\dfrac{1}{\sqrt{x^2+2x+5}}$ to $\dfrac{1}{x}$ with the Limit Comparison Test:

$$\lim_{x\to\infty} \frac{1/\sqrt{x^2+2x+5}}{1/x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+2x+5}}.$$

The immediate evaluation of this limit returns ∞/∞, an indeterminate form. Using l'Hôpital's Rule seems appropriate, but in this situation, it does not lead to useful results. (We encourage the reader to employ l'Hôpital's Rule at least once to verify this.)

The trouble is the square root function. To get rid of it, we employ the following fact: If $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} f(x)^2 = L^2$. (This is true when either c or L is ∞.) So we consider now the limit

$$\lim_{x\to\infty} \frac{x^2}{x^2+2x+5}.$$

This converges to 1, meaning the original limit also converged to 1. As x gets very large, the function $\dfrac{1}{\sqrt{x^2+2x+5}}$ looks very much like $\dfrac{1}{x}$. Since we know that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, by the Limit Comparison Test we know that $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$ also diverges. Figure 7.6.12 graphs $f(x) = 1/\sqrt{x^2+2x+5}$ and $f(x) = 1/x$, il-lustrating that as x gets large, the f unctions become indistinguishable.

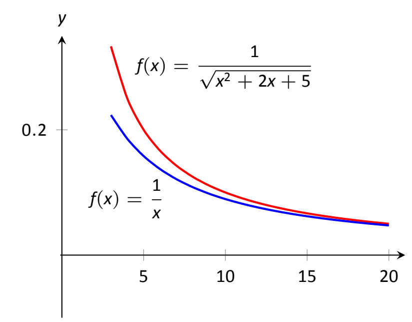


Figure 7.6.12: Graphing $f(x) = \frac{1}{\sqrt{x^2+2x+5}}$ and $f(x) = \frac{1}{x}$ in Example 7.6.6.

Both the Direct and Limit Comparison Tests were given in terms of integrals over an infinite interval. There are versions that apply to improper integrals with an infinite range, but as they are a bit wordy and a little more difficult to employ, they are omitted from this text.

This chapter has explored many integration techniques. We learned Substitution, which "undoes" the Chain Rule of differentiation, as well as Integration by Parts, which "undoes" the Product Rule. We learned specialized techniques for handling trigonometric functions and introduced the hyperbolic functions, which are closely related to the trigonometric functions. All techniques effectively have this goal in common: rewrite the integrand in a new way so that the integration step is easier to see and implement.

As stated before, integration is, in general, hard. It is easy to write a function whose antiderivative is impossible to write in terms of elementary functions, and even when a function does have an antiderivative expressible by elementary functions, it may be really hard to discover what it is. The powerful computer algebra system *Mathematica*® has approximately 1,000 pages of code dedicated to integration.

Do not let this difficulty discourage you. There is great value in learning integration techniques, as they allow one to manipulate an integral in ways that can illuminate a concept for greater understanding. There is also great value in understanding the need for good numerical techniques: the Trapezoidal and Simpson's Rules are just the beginning of powerful techniques for approximating the value of integration.

The next chapter stresses the uses of integration. We generally do not find antiderivatives for antiderivative's sake, but rather because they provide the solution to some type of problem. The following chapter introduces us to a number of different problems whose solution is provided by integration.

Note: The *Apex* author and most mathematicians use the terminology *converge* or *diverge*. We prefer the terms *exists* (*as an extended real number*) or *does not exist* because of concreteness and applications. In applications an infinite answer is always meaningful.

Note: Work a few assigned problem both by the hyperreal method and the limit method. See if there are any for which the hyperreal method does not apply, but for which the limit method works.

Exercises 8.6

Terms and Concepts

1. The definite integral was defined with what two stipulations?

2. If $\lim_{b\to\infty} \int_0^b f(x)\ dx$ exists, then the integral $\int_0^\infty f(x)\ dx$ is said to ________________.

3. If $\int_1^\infty f(x)\ dx = 10$, and $0 \leq g(x) \leq f(x)$ for all x, then we know that $\int_1^\infty g(x)\ dx$ ________________.

4. For what values of p will $\int_1^\infty \frac{1}{x^p}\ dx$ converge?

5. For what values of p will $\int_{10}^\infty \frac{1}{x^p}\ dx$ converge?

6. For what values of p will $\int_0^1 \frac{1}{x^p}\ dx$ converge?

Problems

In Exercises 7 – 34, evaluate the given improper integral.

7. $\int_0^\infty e^{5-2x}\ dx$

8. $\int_1^\infty \frac{1}{x^3}\ dx$

9. $\int_1^\infty x^{-4}\ dx$

10. $\int_{-\infty}^\infty \frac{1}{x^2+9}\ dx$

11. $\int_{-\infty}^0 2^x\ dx$

12. $\int_{-\infty}^0 \left(\frac{1}{2}\right)^x\ dx$

13. $\int_{-\infty}^\infty \frac{x}{x^2+1}\ dx$

14. $\int_3^\infty \frac{1}{x^2-4}\ dx$

15. $\int_2^\infty \frac{1}{(x-1)^2}\ dx$

16. $\int_1^2 \frac{1}{(x-1)^2}\ dx$

17. $\int_2^\infty \frac{1}{x-1}\ dx$

18. $\int_1^2 \frac{1}{x-1}\ dx$

19. $\int_{-1}^1 \frac{1}{x}\ dx$

20. $\int_1^3 \frac{1}{x-2}\ dx$

21. $\int_0^\pi \sec^2 x\ dx$

22. $\int_{-2}^1 \frac{1}{\sqrt{|x|}}\ dx$

23. $\int_0^\infty xe^{-x}\ dx$

24. $\int_0^\infty xe^{-x^2}\ dx$

25. $\int_{-\infty}^\infty xe^{-x^2}\ dx$

26. $\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\ dx$

27. $\int_0^1 x\ln x\ dx$

28. $\int_0^1 x^2\ln x\ dx$

29. $\int_1^\infty \frac{\ln x}{x}\ dx$

30. $\int_0^1 \ln x\ dx$

31. $\int_1^\infty \frac{\ln x}{x^2}\ dx$

32. $\int_1^\infty \frac{\ln x}{\sqrt{x}}\ dx$

33. $\int_0^\infty e^{-x}\sin x\ dx$

34. $\int_0^\infty e^{-x}\cos x\ dx$

In Exercises 35 – 44, use the Direct Comparison Test or the Limit Comparison Test to determine whether the given definite integral converges or diverges. Clearly state what test is being used and what function the integrand is being compared to.

35. $\displaystyle\int_{10}^{\infty} \frac{3}{\sqrt{3x^2+2x-5}}\,dx$

36. $\displaystyle\int_{2}^{\infty} \frac{4}{\sqrt{7x^3-x}}\,dx$

37. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x+3}}{\sqrt{x^3-x^2+x+1}}\,dx$

38. $\displaystyle\int_{1}^{\infty} e^{-x}\ln x\,dx$

39. $\displaystyle\int_{5}^{\infty} e^{-x^2+3x+1}\,dx$

40. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{e^x}\,dx$

41. $\displaystyle\int_{2}^{\infty} \frac{1}{x^2+\sin x}\,dx$

42. $\displaystyle\int_{0}^{\infty} \frac{x}{x^2+\cos x}\,dx$

43. $\displaystyle\int_{0}^{\infty} \frac{1}{x+e^x}\,dx$

44. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x-x}\,dx$

Solutions 7.7

1. $0/0, \infty/\infty, 0\cdot\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$
2. F
3. F
4. The base of an expression is approaching 1 while its power is growing without bound.
5. derivatives; limits
6. Answers will vary.
7. Answers will vary.
8. Answers will vary.
9. 3
10. $-5/3$
11. -1
12. $-\sqrt{2}/2$
13. 5
14. 0
15. $2/3$
16. a/b
17. ∞
18. $1/2$
19. 0
20. 0
21. 0
22. ∞
23. ∞
24. ∞
25. 0
26. 2
27. -2
28. 0
29. 0
30. 0
31. 0
32. 0
33. ∞
34. ∞
35. ∞
36. 0
37. 0
38. e
39. 1
40. 1
41. 1
42. 1
43. 1
44. 0
45. 1
46. 1
47. 1
48. 1
49. 2
50. $1/2$
51. $-\infty$
52. 1
53. 0
54. 3

7.7 Numerical Integration

The Fundamental Theorem of Calculus gives a concrete technique for finding the exact value of a definite integral. That technique is based on computing antiderivatives. Despite the power of this theorem, there are still situations where we must *approximate* the value of the definite integral instead of finding its exact value. The first situation we explore is where we *cannot* compute the antiderivative of the integrand. The second case is when we actually do not know the function in the integrand, but only its value when evaluated at certain points.

An **elementary function** is any function that is a combination of polynomial, n^{th} root, rational, exponential, logarithmic and trigonometric functions. We can compute the derivative of any elementary function, but there are many elementary functions of which we cannot compute an antiderivative. For example, the following functions do not have antiderivatives that we can express with elementary functions:

$$e^{-x^2}, \quad \sin(x^3) \quad \text{and} \quad \frac{\sin x}{x}.$$

The simplest way to refer to the antiderivatives of e^{-x^2} is to simply write $\int e^{-x^2}\,dx$.

This section outlines three common methods of approximating the value of definite integrals. We describe each as a systematic method of approximating area under a curve. By approximating this area accurately, we find an accurate approximation of the corresponding definite integral.

We will apply the methods we learn in this section to the following definite integrals:

$$\int_0^1 e^{-x^2}\,dx, \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx, \quad \text{and} \quad \int_{0.5}^{4\pi} \frac{\sin(x)}{x}\,dx,$$

as pictured in Figure 7.7.1.

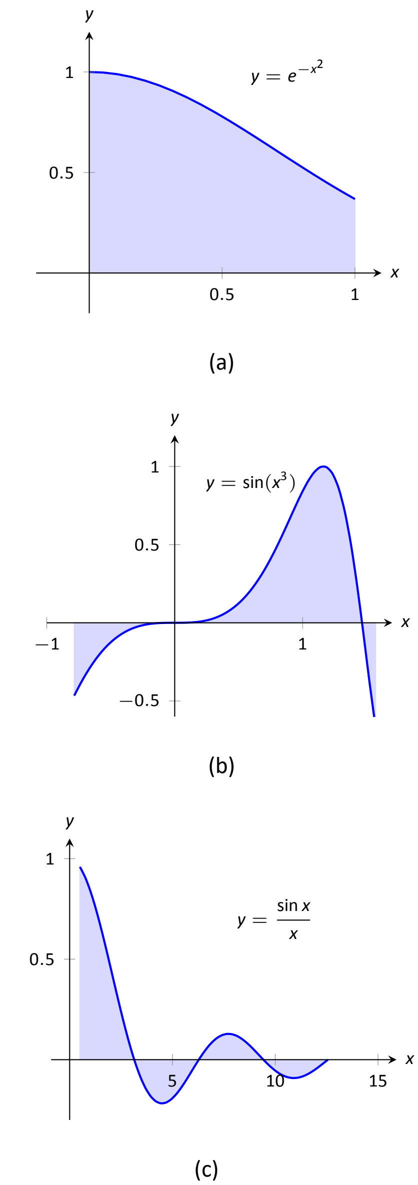


Figure 7.7.1: Graphically representing three definite integrals that cannot be evaluated using antiderivatives.

The Left and Right Hand Rule Methods Earlier we addressed the problem of evaluating definite integrals by approximating the area under the curve using rectangles. We revisit those ideas here before introducing other methods of approximating definite integrals.

We start with a review of notation. Let f be a continuous function on the interval $[a, b]$. We wish to approximate $\displaystyle\int_a^b f(x)\,dx$. We partition $[a, b]$ into n equally spaced subintervals, each of length $\Delta x = \dfrac{b-a}{n}$. The endpoints of these subintervals are labeled as

$$x_1 = a,\ x_2 = a + \Delta x,\ x_3 = a + 2\Delta x,\ \ldots,\ x_i = a + (i-1)\Delta x,\ \ldots,\ x_{n+1} = b.$$

Key Idea 5.3.1 states that to use the Left Hand Rule we use the summation $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$ and to use the Right Hand Rule we use $\displaystyle\sum_{i=1}^{n} f(x_{i+1})\Delta x$. We review the use of these rules in the context of examples.

Example 7.7.1 Approximating definite integrals with rectangles

Approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using the Left and Right Hand Rules with 5 equally spaced subintervals.

Solution We begin by partitioning the interval $[0,1]$ into 5 equally spaced intervals. We have $\Delta x = \frac{1-0}{5} = 1/5 = 0.2$, so

$$x_1 = 0,\ x_2 = 0.2,\ x_3 = 0.4,\ x_4 = 0.6,\ x_5 = 0.8,\ \text{and } x_6 = 1.$$

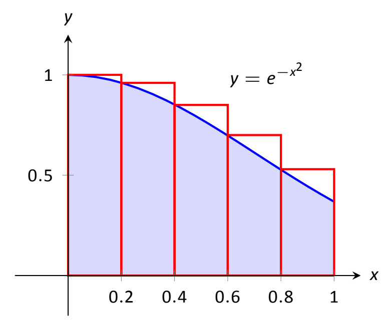
Using the Left Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \big(f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\big)\Delta x\\ &= \big(f(0)+f(0.2)+f(0.4)+f(0.6)+f(0.8)\big)\Delta x\\ &\doteq \big(1+0.961+0.852+0.698+0.527\big)(0.2)\\ &\doteq 0.808.\end{aligned}$$

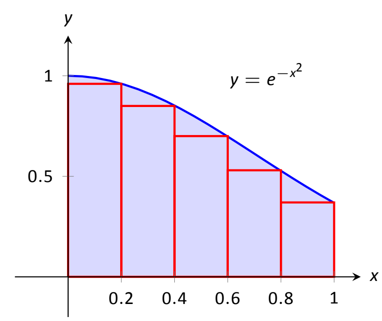
Using the Right Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_{i+1})\Delta x &= \big(f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)\big)\Delta x\\ &= \big(f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1)\big)\Delta x\\ &\doteq \big(0.961+0.852+0.698+0.527+0.368\big)(0.2)\\ &\doteq 0.681.\end{aligned}$$

Figure 7.7.2 shows the rectangles used in each method to approximate the definite integral. These graphs show that in this particular case, the Left Hand Rule is an over approximation and the Right Hand Rule is an under approximation. To get a better approximation, we could use more rectangles, as we did



(a)



(b)

Figure 7.7.2: Approximating $\int_0^1 e^{-x^2}\,dx$ in Example 7.7.1.

x_i	Exact	Approx.	$\sin(x_i^3)$
x_1	$-\pi/4$	-0.785	-0.466
x_2	$-7\pi/40$	-0.550	-0.165
x_3	$-\pi/10$	-0.314	-0.031
x_4	$-\pi/40$	-0.0785	0
x_5	$\pi/20$	0.157	0.004
x_6	$\pi/8$	0.393	0.061
x_7	$\pi/5$	0.628	0.246
x_8	$11\pi/40$	0.864	0.601
x_9	$7\pi/20$	1.10	0.971
x_{10}	$17\pi/40$	1.34	0.690
x_{11}	$\pi/2$	1.57	-0.670

Figure 7.7.3: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\; dx$.

earlier. We could also average the Left and Right Hand Rule results together, giving

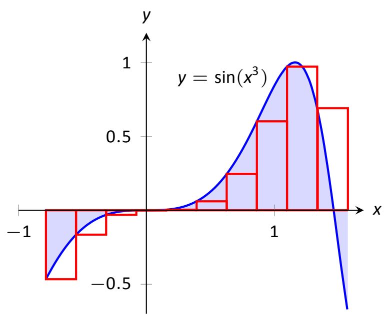
$$\frac{0.808 + 0.681}{2} = 0.7445.$$

Solution We begin by finding Δx:

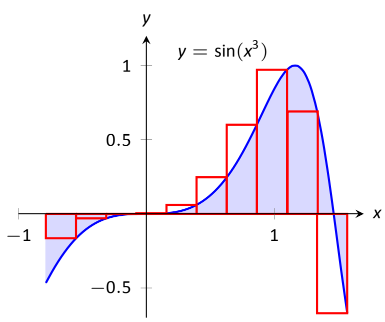
$$\frac{b-a}{n} = \frac{\pi/2 - (-\pi/4)}{10} = \frac{3\pi}{40} \doteq 0.236.$$

It is useful to write out the endpoints of the subintervals in a table; in Figure 7.7.3, we give the exact values of the endpoints, their decimal approximations, and decimal approximations of $\sin(x^3)$ evaluated at these points.

Once this table is created, it is straightforward to approximate the definite integral using the Left and Right Hand Rules. (Note: the table itself is easy to create, especially with a standard spreadsheet program on a computer. The last two columns are all that are needed.) The Left Hand Rule sums the first 10 values of $\sin^3 x$) and multiplies the sum by Δx; the Right Hand Rule sums the last 10.



(a)



(b)

Figure 7.7.4 Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx$ in Example 7.7.2.

Left Hand Rule: $\displaystyle {}_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx \doteq (1.91)(0.236) = 0.451.$

Right Hand Rule: $\displaystyle \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx \doteq (1.71)(0.236) = 0.404.$

Average of the Left and Right Hand Rules: 0.4275.

The actual answer, accurate to 3 places after the decimal, is 0.460. Our approximations were once again fairly good. The rectangles used in each approximation are shown in Figure 7.7.4. It is clear from the graphs that using more rectangles (and hence, narrower rectangles) should result in a more accurate approximation.

The Trapezoidal Rule

In Example 7.7.1 we approximated the value of $\int_0^1 e^{-x^2}\, dx$ with 5 rectangles of equal width. Figure 7.7.2 shows the rectangles used in the Left and Right

Hand Rules. These graphs clearly show that rectangles do not match the shape of the graph all that well, and that accurate approximations will only come by using lots of rectangles.

Instead of using rectangles to approximate the area, we can instead use *trapezoids.* In Figure 5.5.5, we show the region under $f(x) = e^{-x^2}$ on $[0, 1]$ approximated with 5 trapezoids of equal width; the top "corners" of each trapezoid lies on the graph of $f(x)$. It is clear from this figure that these trapezoids more accurately approximate the area under f and hence should give a better approximation of $\int_0^1 e^{-x^2}\,dx$. (In fact, these trapezoids seem to give a *great* approximation of the area!)

The formula for the area of a trapezoid is given in Figure 5.5.6. We approximate $\int_0^1 e^{-x^2}\,dx$ with these trapezoids in the following example.

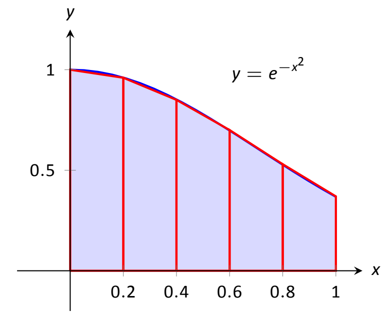


Figure 7.7.5: Approximating $\int_0^1 e^{-x^2}\,dx$ using 5 trapezoids of equal widths.

Example 7.7.3 Approximating definite integrals using trapezoids

Use 5 trapezoids of equal width to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution To compute the areas of the 5 trapezoids in Figure 5.5.5, it will again be useful to create a table of values as shown in Figure 5.5.7.

The leftmost trapezoid has legs of length 1 and 0.961 and a height of 0.2. Thus, by our formula, the area of the leftmost trapezoid is:

$$\frac{1+0.961}{2}(0.2) = 0.1961.$$

Moving right, the next trapezoid has legs of length 0.961 and 0.852 and a height of 0.2. Thus its area is:

$$\frac{0.961+0.852}{2}(0.2) = 0.1813.$$

The sum of the areas of all 5 trapezoids is:

$$\frac{1+0.961}{2}(0.2) + \frac{0.961+0.852}{2}(0.2) + \frac{0.852+0.698}{2}(0.2) + \frac{0.698+0.527}{2}(0.2) + \frac{0.527+0.368}{2}(0.2) = 0.7445.$$

We approximate $\displaystyle\int_0^1 e^{-x^2}\,dx \doteq 0.7445$.

There are many things to observe in this example. Note how each term in the final summation was multiplied by both 1/2 and by $\Delta x = 0.2$. We can factor these coefficients out, leaving a more concise summation as:

$$\frac{1}{2}(0.2)\Big[(1+0.961)+(0.961+0.852)+(0.852+0.698)+(0.698+0.527)+(0.527+0.368)\Big].$$

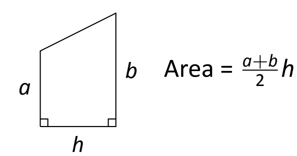


Figure 7.7.6: The area of a trapezoid.

x_i	$e^{-x_i^2}$
0	1
0.2	0.961
0.4	0.852
0.6	0.698
0.8	0.527
1	0.368

Figure 7.77: A table of values of e^{-x^2}.

Now notice that all numbers except for the first and the last are added twice. Therefore we can write the summation even more concisely as

$$\frac{0.2}{2}\Big[1 + 2(0.961 + 0.852 + 0.698 + 0.527) + 0.368\Big].$$

This is the heart of the **Trapezoidal Rule**, wherein a definite integral $\int_a^b f(x)\,dx$ is approximated by using trapezoids of equal widths to approximate the corresponding area under f. Using n equally spaced subintervals with endpoints x_1, x_2, ..., x_{n+1}, we again have $\Delta x = \dfrac{b-a}{n}$. Thus:

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \sum_{i=1}^{n} \frac{f(x_i)+f(x_{i+1})}{2}\Delta x\\ &= \frac{\Delta x}{2}\sum_{i=1}^{n}\big(f(x_i)+f(x_{i+1})\big)\\ &= \frac{\Delta x}{2}\Big[f(x_1) + 2\sum_{i=2}^{n} f(x_i) + f(x_{n+1})\Big].\end{aligned}$$

Example 7.7.4 Using the Trapezoidal Rule

Revisit Example 7.7.2 and approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Trapezoidal Rule and 10 equally spaced subintervals.

SOLUTION We refer back to Figure 7.7.3 for the table of values of $\sin(x^3)$. Recall that $\Delta x = 3\pi/40 \doteq 0.236$. Thus we have:

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx &\doteq \frac{0.236}{2}\Big[-0.466 + 2\Big(-0.165 + (-0.031) + \ldots + 0.69\Big) + (-0.67)\Big]\\ &= 0.4275.\end{aligned}$$

Notice how "quickly" the Trapezoidal Rule can be implemented once the table of values is created. This is true for all the methods explored in this section; the real work is creating a table of x_i and $f(x_i)$ values. Once this is completed, approximating the definite integral is not difficult. Again, using technology is wise. Spreadsheets can make quick work of these computations and make using lots of subintervals easy.

Also notice the approximations the Trapezoidal Rule gives. It is the average of the approximations given by the Left and Right Hand Rules! This effectively renders the Left and Right Hand Rules obsolete. They are useful when first learning about definite integrals, but if a real approximation is needed, one is generally better off using the Trapezoidal Rule instead of either the Left or Right Hand Rule.

How can we improve on the Trapezoidal Rule, apart from using more and more trapezoids? The answer is clear once we look back and consider what we have *really* done so far. The Left Hand Rule is not *really* about using rectangles to approximate area. Instead, it approximates a function f with constant functions on small subintervals and then computes the definite integral of these constant functions. The Trapezoidal Rule is really approximating a function f with a linear function on a small subinterval, then computes the definite integral of this linear function. In both of these cases the definite integrals are easy to compute in geometric terms.

So we have a progression: we start by approximating f with a constant function and then with a linear function. What is next? A quadratic function. By approximating the curve of a function with lots of parabolas, we generally get an even better approximation of the definite integral. We call this process **Simpson's Rule**, named after Thomas Simpson (1710-1761), even though others had used this rule as much as 100 years prior.

Simpson's Rule

Given one point, we can create a constant function that goes through that point. Given two points, we can create a linear function that goes through those points. Given three points, we can create a quadratic function that goes through those three points (given that no two have the same x–value).

Consider three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) whose x–values are equally spaced and $x_1 < x_2 < x_3$. Let f be the quadratic function that goes through these three points. It is not hard to show that

$$\int_{x_1}^{x_3} f(x)\,dx = \frac{x_3 - x_1}{6}\big(y_1 + 4y_2 + y_3\big). \tag{7.4}$$

Consider Figure 5.5.8. A function f goes through the 3 points shown and the parabola g that also goes through those points is graphed with a dashed line. Using our equation from above, we know exactly that

$$\int_1^3 g(x)\,dx = \frac{3-1}{6}\big(3 + 4(1) + 2\big) = 3.$$

Since g is a good approximation for f on $[1, 3]$, we can state that

$$\int_1^3 f(x)\,dx \doteq 3.$$

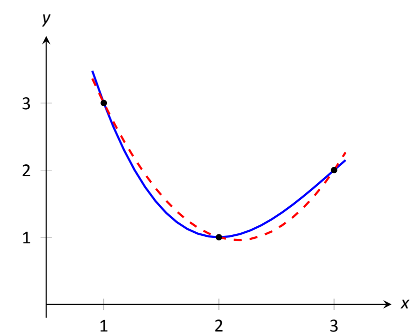
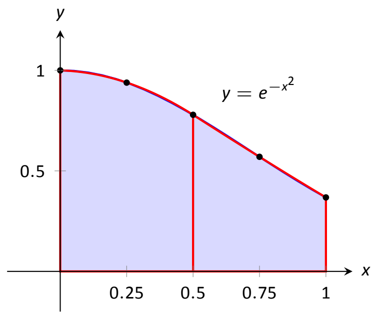


Figure 7.7.8: A graph of a function f and a parabola that approximates it well on $[1, 3]$.

x_i	$e^{-x_i^2}$
0	1
0.25	0.939
0.5	0.779
0.75	0.570
1	0.368

(a)



(b) Figure 7.7.9: A table of values

Notice how the interval $[1, 3]$ was split into two subintervals as we needed 3 points. Because of this, whenever we use Simpson's Rule, we need to break the interval into an even number of subintervals.

In general, to approximate $\int_a^b f(x)\,dx$ using Simpson's Rule, subdivide $[a, b]$ into n subintervals, where n is even and each subinterval has width $\Delta x = (b - a)/n$. We approximate f with $n/2$ parabolic curves, using Equation (5.4) to compute the area under these parabolas. Adding up these areas gives the formula:

$$\int_a^b f(x)\,dx \doteq \frac{\Delta x}{3}\Big[f(x_1)+4f(x_2)+2f(x_3)+4f(x_4)+\ldots+2f(x_{n-1})+4f(x_n)+f(x_{n+1})\Big].$$

Note how the coefficients of the terms in the summation have the pattern 1, 4, 2, 4, 2, 4, . . ., 2, 4, 1.

Let's demonstrate Simpson's Rule with a concrete example.

Example 7.7.5 Using Simpson's Rule

Approximate $\int_0^1 e^{-x^2}\,dx$ using Simpson's Rule and 4 equally spaced subintervals.

Solution We begin by making a table of values as we have in the past, as shown in Figure 5.5.9(a). Simpson's Rule states that

$$\int_0^1 e^{-x^2}\,dx \doteq \frac{0.25}{3}\Big[1 + 4(0.939) + 2(0.779) + 4(0.570) + 0.368\Big] = 0.7468\overline{3}.$$

Recall in Example 7.7.1 we stated that the correct answer, accurate to 4 places after the decimal, was 0.7468. Our approximation with Simpson's Rule, with 4 subintervals, is better than our approximation with the Trapezoidal Rule using 5!

Figure 7.7.9(b) shows $f(x) = e^{-x^2}$ along with its approximating parabolas, demonstrating how good our approximation is. The approximating curves are nearly indistinguishable from the actual function.

x_i	$\sin(x_i^3)$
-0.785	-0.466
-0.550	-0.165
-0.314	-0.031
-0.0785	0
0.157	0.004
0.393	0.061
0.628	0.246
0.864	0.601
1.10	0.971
1.34	0.690
1.57	-0.670

Figure 5.5.10: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 5.5.6.

Example 7.7.6 Using Simpson's Rule

Approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using Simpson's Rule and 10 equally spaced intervals.

Solution Figure 5.5.10 shows the table of values that we used in the past for this problem, shown here again for convenience. Again, $\Delta x = (\pi/2 + \pi/4)/10 \doteq 0.236$.

Simpson's Rule states that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \doteq \frac{0.236}{3}\Big[(-0.466) + 4(-0.165) + 2(-0.031) + \ldots$$
$$\ldots + 2(0.971) + 4(0.69) + (-0.67)\Big]$$
$$= 0.4701$$

Recall that the actual value, accurate to 3 decimal places, is 0.460. Our approximation is within one $1/100^{\text{th}}$ of the correct value. The graph in Figure 7.7.11 shows how closely the parabolas match the shape of the graph.

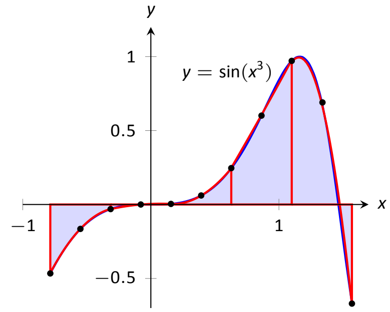


Figure 5.5.11: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ in Example 5.5.6 with Simpson's Rule and 10 equally spaced intervals.

Summary

We summarize the key concepts of this section thus far in the following Key Idea.

Key Idea 7.7.1 Numerical Integration

Let f be a continuous function on $[a, b]$, let n be a positive integer, and let $\Delta x = \dfrac{b-a}{n}$.
Set $x_1 = a$, $x_2 = a + \Delta x, \ldots, x_i = a + (i-1)\Delta x$, $x_{n+1} = b$.

Consider $\displaystyle\int_a^b f(x)\,dx$.

Left Hand Rule: $\displaystyle\int_a^b f(x)\,dx \doteq \Delta x\Big[f(x_1) + f(x_2) + \ldots + f(x_n)\Big]$.

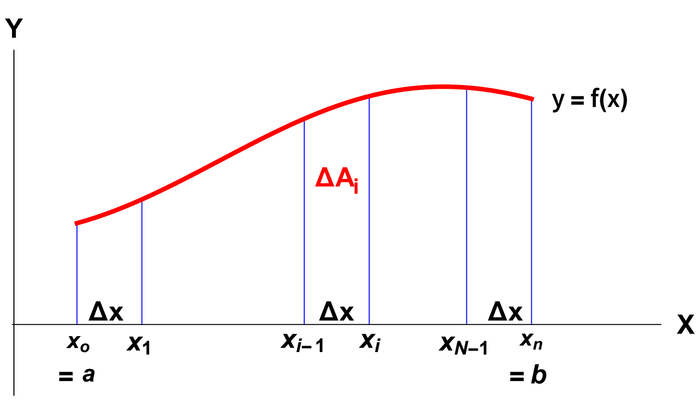
Right Hand Rule: $\displaystyle\int_a^b f(x)\,dx \doteq \Delta x\Big[f(x_2) + f(x_3) + \ldots + f(x_{n+1})\Big]$.

Trapezoidal Rule: $\displaystyle\int_a^b f(x)\,dx \doteq \frac{\Delta x}{2}\Big[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\Big]$.

Simpson's Rule: $\displaystyle\int_a^b f(x)\,dx \doteq \frac{\Delta x}{3}\Big[f(x_1) + 4f(x_2) + 2f(x_3) + \ldots + 4f(x_n) + f(x_{n+1})\Big]$ (n even).

You probably <G FGL @9N= LG ?=L N=JQ 9L FME=JA;9D AFL=?J9LAGFKj !GEHML=JK ;9F <G 9 FME=JA;9D AFL=?J9LAGF LG 9K E9FQ <=;AE9D HD9;=K 9K QGM OAK@j 2@=J= AK 9 >J== GFRDAF= KAL=, Wolfram Alpha, for example, which will do numerical integration. Also, any professional CAS will do these.

For programmable calculators or computers - simply sum each term below from 1 to n for every rectangle. The formulas above are for hand calculations. Hopefully you do this more than once.



LH Rule $\Delta A_i = \Delta x\ y_{i-1}$

RH Rule $\Delta A_i = \Delta x\, y_i$

Trapezoidal Rule $\Delta A_i = \dfrac{\Delta x}{2}(y_{i-1} + y_i)$

Simpson's Rule $\Delta A_i = \dfrac{\Delta x}{3}(y_{i-1} + 4\,y_i + y_{i+1})$

Exercises 7.7

Terms and Concepts

1. T/F: Simpson's Rule is a method of approximating antiderivatives.

2. What are the two basic situations where approximating the value of a definite integral is necessary?

3. Why are the Left and Right Hand Rules rarely used?

4. Simpson's Rule is based on approximating portions of a function with what type of function?

Problems

In Exercises 5 – 12, a definite integral is given.

(a) Approximate the definite integral with the Trapezoidal Rule and $n = 4$.

(b) Approximate the definite integral with Simpson's Rule and $n = 4$.

(c) Find the exact value of the integral.

5. $\int_{-1}^{1} x^2\, dx$

6. $\int_{0}^{10} 5x\, dx$

7. $\int_{0}^{\pi} \sin x\, dx$

8. $\int_{0}^{4} \sqrt{x}\, dx$

9. $\int_{0}^{3} (x^3 + 2x^2 - 5x + 7)\, dx$

10. $\int_{0}^{1} x^4\, dx$

11. $\int_{0}^{2\pi} \cos x\, dx$

12. $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$

In Exercises 13 – 20, approximate the definite integral with the Trapezoidal Rule and Simpson's Rule, with $n = 6$.

13. $\int_{0}^{1} \cos\left(x^2\right)\, dx$

14. $\int_{-1}^{1} e^{x^2}\, dx$

15. $\int_{0}^{5} \sqrt{x^2 + 1}\, dx$

16. $\int_{0}^{\pi} x \sin x\, dx$

17. $\int_{0}^{\pi/2} \sqrt{\cos x}\, dx$

18. $\int_{1}^{4} \ln x\, dx$

19. $\int_{-1}^{1} \frac{1}{\sin x + 2}\, dx$

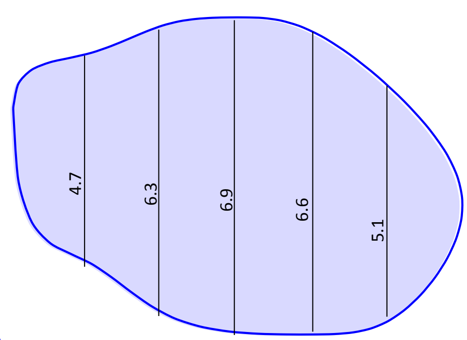
20. $\int_{0}^{6} \frac{1}{\sin x + 2}\, dx$

In Exercises 25 – 26, a region is given. Find the area of the region using both the Trapezoid's and Simpson's Rule:

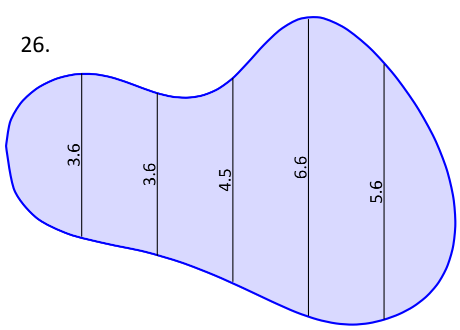
(a) where the measurements are in centimeters, taken in 1 cm increments, and

(b) where the measurements are in hundreds of yards, taken in 100 yd increments.

25.



26.



NOTE #25 and #26 are especially good questions if you are going to build a free form swimming pool.

Solutions 7.7

1. F
2. When the antiderivative cannot be computed and when the integrand is unknown.
3. They are superseded by the Trapezoidal Rule; it takes an equal amount of work and is generally more accurate.
4. A quadratic function (i.e., parabola)
5. (a) $3/4$
   (b) $2/3$
   (c) $2/3$
6. (a) 250
   (b) 250
   (c) 250
7. (a) $\frac{1}{4}(1+\sqrt{2})\pi \doteq 1.896$
   (b) $\frac{1}{6}(1+2\sqrt{2})\pi \doteq 2.005$
   (c) 2
8. (a) $2+\sqrt{2}+\sqrt{3} \doteq 5.15$
   (b) $2/3(3+\sqrt{2}+2\sqrt{3}) \doteq 5.25$
   (c) $16/3 \doteq 5.33$
9. (a) 38.5781
   (b) $147/4 \doteq 36.75$
   (c) $147/4 \doteq 36.75$
10. (a) 0.2207
    (b) 0.2005
    (c) $1/5$
11. (a) 0
    (b) 0
    (c) 0
12. (a) $9/2(1+\sqrt{3}) \doteq 12.294$
    (b) $3+6\sqrt{3} \doteq 13.392$
    (c) $9\pi/2 \doteq 14.137$
13. Trapezoidal Rule: 0.9006
    Simpson's Rule: 0.90452
14. Trapezoidal Rule: 3.0241
    Simpson's Rule: 2.9315
15. Trapezoidal Rule: 13.9604
    Simpson's Rule: 13.9066
16. Trapezoidal Rule: 3.0695
    Simpson's Rule: 3.14295
17. Trapezoidal Rule: 1.1703
    Simpson's Rule: 1.1873
18. Trapezoidal Rule: 2.52971
    Simpson's Rule: 2.5447
19. Trapezoidal Rule: 1.0803
    Simpson's Rule: 1.077
20. Trapezoidal Rule: 3.5472
    Simpson's Rule: 3.6133

Chapter 8 Applications of Integration

Preview Differentials are important in discovering the fundamental law governing a quantity Q; over a short interval its behavior may be quite simple. If you want to know its growth rate, you divide by dt or perhaps dx depending on whether the quantity changes in time or space. In the applications of this chapter, the total amount or the change in the amount Q is desired; it is the integral of dQ obtained by summing the differentials of Q obtaining $\int_{t_1}^{t_2} dQ$ or $\int_{x_1}^{x_2} dQ$.

In this chapter we examine the process of first determining the differential law and then obtaining its integral, the total amount. We will call this the ***Five Step Procedure***. The steps are:

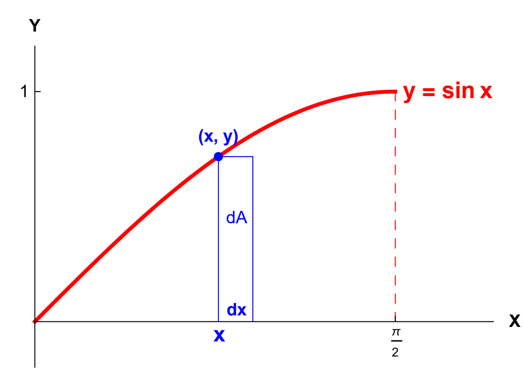
I. Draw a picture illustrating the quantity. Label all quantities used in the problem.
II. Show a typical differential region. Label.
III. Find a simple formula for the differential element which is asymptotically equal its exact amount.
IV. Integrate.
V. Evaluate.

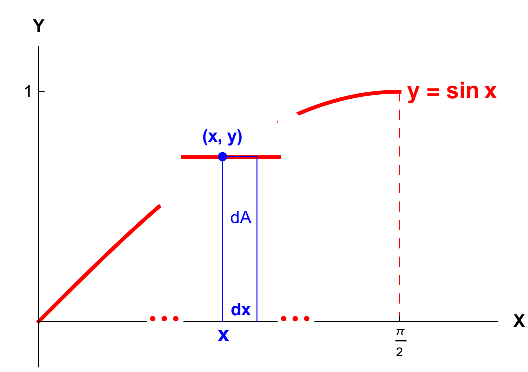
This procedure should be used on all relevant exercises in this chapter. Using end integral formulas often lead to mistakes. Also you want to be fluent with this procedure so that you can readily use it in your area of application. In the the sections from Apex Calculus, five steps are also required. In exams each step is worth 20% of the total points awarded the problem.

Note that we will assume that the bounding functions are continuous so that dx an infinitesimal implies dy is an infinitesimal. We will begin with an example from Chapter 5, finding the area under a curve.

Example Find the area under the curve $y = \sin x, 0 \le x \le \frac{\pi}{2}$.

I, II





III

Method 1

Sin x is continuous. So dx an infinitesimal implies dy an infinitesimal.

$$dA \approx \sin(x)dx$$

Here, because the curve is continuous, dx an infinitesimal implies dy is an infinitesimal which implies an ≈ is justified.

Method 2

Properly magnified, sin x appears constant, so

$$dA \approx \sin(x)dx$$

Here, because the element looks exact, implies an ≈ is justified. In applications such intuitive reasoning is often justified.

$$\textbf{IV} \quad A = \int_0^{\pi/2} \sin x \, dx$$

$$\textbf{V} \quad = -\cos x \Big|_0^{\pi/2}$$

$$= 1.$$

8.1 Area Between Curves

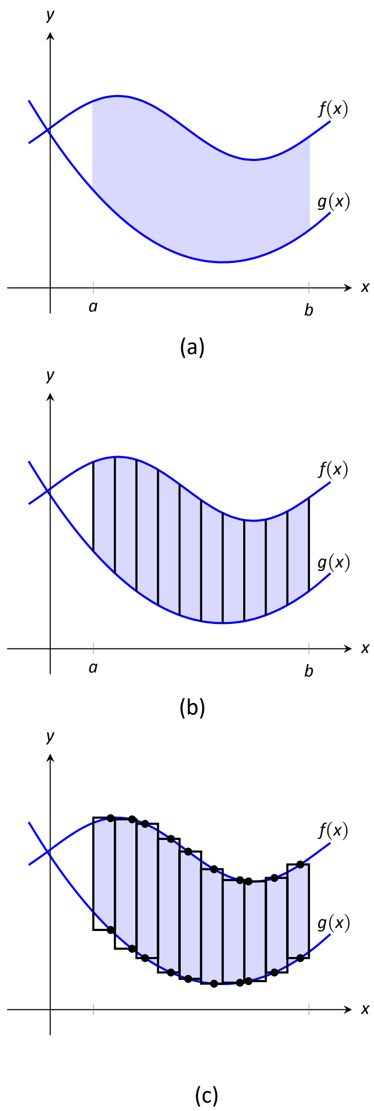


Figure 8.1.1: Subdividing a region into vertical slices and approximating the areas with rectangles.

We are often interested in knowing the area of a region. Forget momentarily that we addressed this already in Section 5.4 and approach it instead using the technique described in Key Idea 8.0.1.

Let Q be the area of a region bounded by continuous functions f and g. If we break the region into many subregions, we have an obvious equation:

Total Area = sum of the areas of the subregions.

The issue to address next is how to systematically break a region into subregions. A graph will help. Consider Figure 8.1.1 (a) where a region between two curves is shaded. While there are many ways to break this into subregions, one particularly efficient way is to "slice" it vertically, as shown in Figure 8.1.1 (b), into n equally spaced slices.

We now approximate the area of a slice. Again, we have many options, but using a rectangle seems simplest. Picking any x-value c_i in the i-th slice, we set

the height of the rectangle to be $f(c_i) - g(c_i)$, the difference of the corresponding y-values. The width of the rectangle is a small difference in x-values, which we represent with Δx. Figure 8.1.1(c) shows sample points c_i chosen in each subinterval and appropriate rectangles drawn. (Each of these rectangles represents a differential element.) Each slice has an area approximately equal to $[f(c_i) - g(c_i)]\,\Delta x$; hence, the total area is approximately the Riemann Sum

$$Q = \sum_{i=1}^{n} \big(f(c_i) - g(c_i)\big)\Delta x.$$

Taking the limit as $n \to \infty$ gives the exact area as $\int_a^b \big(f(x) - g(x)\big)\,dx$.

Theorem 8.1.1 Area Between Curves (restatement of Theorem 5.4.3)

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Note again from the above graphs that clearly

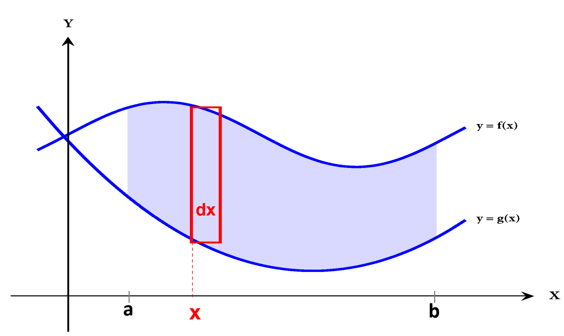
$dA \approx$ [top - bottom]dx

$= [f(x) - g(x)]dx.$

Integrating from a to b gives the area. The text argument is complicated, hard to remember and no more rigorous.
(But still, sometimes a detailed discussion is useful.)

I

II



bh¢9 Usually only one *typical differential element* [illegible] **width dx should also be shown.**

In exercises and exams always use the full *Five Step Procedure*.

III $\quad dA \approx [f(x) - g(x)]dx$

IV $\quad A = \int_a^b [f(x) - g(x)]\,dx$

V $\quad = F(x) - G(x)\Big]_a^b$

Example 7.1.1 Finding area enclosed by curves

Find the area of the region bounded by $f(x) = \sin x + 2$, $g(x) = -\frac{1}{2}\cos(2x) - 1$ $x = 0$ and $x = 4\pi$, as shown in Figure 8.1.2.

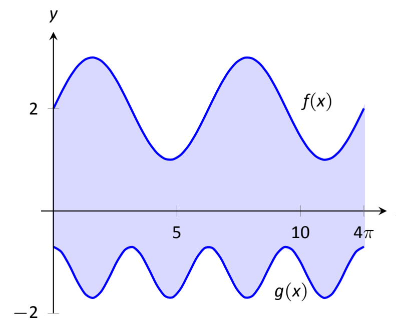


Figure 8.1.2: Graphing an enclosed region in Example 8.1.1.

SOLUTION The graph verifies that the upper boundary of the region is given by f and the lower bound is given by g. Therefore the area of the region is the value of the integral

$$\begin{aligned}\int_0^{4\pi} \big(f(x) - g(x)\big)\,dx &= \int_0^{4\pi} \Big(\sin x + 2 - \big(\frac{1}{2}\cos(2x) - 1\big)\Big)\,dx \\ &= -\cos x - \frac{1}{4}\sin(2x) + 3x\Big|_0^{4\pi} \\ &= 12\pi \doteq 37.7 \text{ units}^2.\end{aligned}$$

Example 8.1.2 Finding total area enclosed by curves

Find the total area of the region enclosed by the functions $f(x) = -2x + 5$ and $g(x) = x^3 - 7x^2 + 12x - 3$ as shown in Figure 8.1.3.

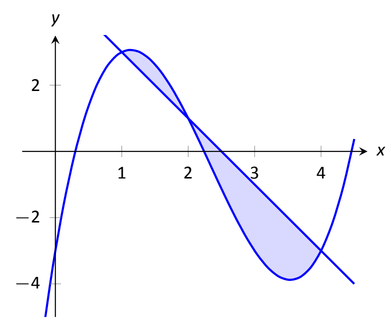


Figure 8.1.3: Graphing a region enclosed by two functions in Example 7.1.2.

SOLUTION A quick calculation shows that $f = g$ at $x = 1, 2$ and 4. One can proceed thoughtlessly by computing $\displaystyle\int_1^4 \big(f(x) - g(x)\big)\,dx$, but this ignores the fact that on $[1, 2]$, $g(x) > f(x)$. (In fact, the thoughtless integration returns $-9/4$, hardly the expected value of an *area*.) Thus we compute the total area by breaking the interval $[1, 4]$ into two subintervals, $[1, 2]$ and $[2, 4]$ and using the proper integrand in each.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big(g(x) - f(x)\big)\,dx + \int_2^4 \big(f(x) - g(x)\big)\,dx \\ &= \int_1^2 \big(x^3 - 7x^2 + 14x - 8\big)\,dx + \int_2^4 \big(-x^3 + 7x^2 - 14x + 8\big)\,dx \\ &= 5/12 + 8/3 \\ &= 37/12 = 3.083 \text{ units}^2.\end{aligned}$$

The previous example makes note that we are expecting area to be *positive*. When first learning about the definite integral, we interpreted it as "signed area under the curve," allowing for "negative area." That doesn't apply here; area is to be positive.

The previous example also demonstrates that we often have to break a given region into subregions before applying Theorem 8.1.1. The following example shows another situation where this is applicable, along with an alternate view of applying the Theorem.

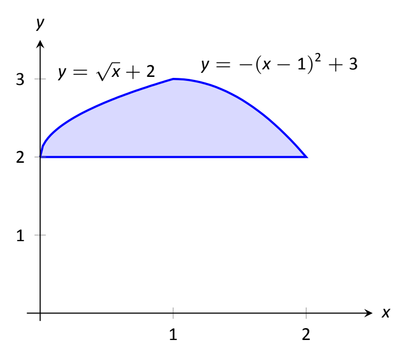


Figure 8.1.4: Graphing a region for Example 8.1.3.

Example 8.1.3 Finding area: integrating with respect to y

Find the area of the region enclosed by the functions $y = \sqrt{x} + 2$, $y = -(x-1)^2 + 3$ and $y = 2$, as shown in Figure 8.1.4.

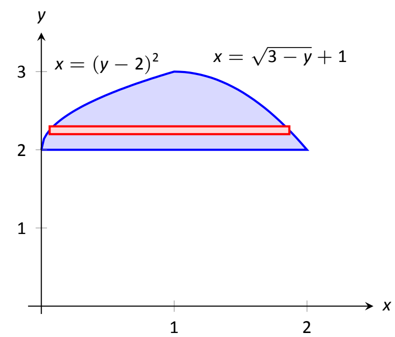


Figure 8.1.5: The region used in Example 8.1.3 with boundaries relabeled as func-tions of y.

Solution We give two approaches to this problem. In the first approach, we notice that the region's "right - left" is defined by two different curves. On $[0, 1]$, the top function is $y=\sqrt{x}+2$; on $[1, 2]$, the top function is $y=-(x-1)^2+3$. Thus we compute the area as the sum of two integrals:

$$\begin{aligned}\text{Total Area} &= \int_0^1 \Big(\big(\sqrt{x}+2\big)-2\Big)\,dx + \int_1^2 \Big(\big(-(x-1)^2+3\big)-2\Big)\,dx\\ &= 2/3+2/3\\ &= 4/3.\end{aligned}$$

The second approach is clever and very useful in certain situations. We are used to viewing curves as functions of x; we input an x-value and a y-value is returned. Some curves can also be described as functions of y: input a y-value and an x-value is returned. We can rewrite the equations describing the boundary by solving for x:

$$y=\sqrt{x}+2 \Rightarrow x=(y-2)^2,\quad y=-(x-1)^2+3 \Rightarrow x=3-\sqrt{y+1}\,.$$

Figure 8.1.5 shows the region with the boundaries relabeled. A differential element, a horizontal rectangle, is also pictured. The width of the rectangle is a small change in y: Δy. The height of the rectangle is a difference in x-values. The "top" x-value is the largest value, i.e., the rightmost. The "bottom" x-value is the smaller, i.e., the leftmost. Therefore the height of the rectangle is

$$\big(\sqrt{3-y}+1\big)-(y-2)^2.$$

The area is found by integrating the above function with respect to y with the appropriate bounds. We determine these by considering the y-values the region occupies. It is bounded below by $y=2$, and bounded above by $y=3$. That is, both the "top" and "bottom" functions exist on the y interval $[2, 3]$. Thus

$$\begin{aligned}\text{Total Area} &= \int_2^3 \big(\sqrt{3-y}+1-(y-2)^2\big)\,dy\\ &= \left(-\frac{2}{3}(3-y)^{3/2}+y-\frac{1}{3}(y-2)^3\right)\Bigg|_2^3\\ &= 4/3.\end{aligned}$$

This calculus–based technique of finding area can be useful even with shapes that we normally think of as "easy." Example 8.1.4 computes the area of a triangle. While the formula "$\frac{1}{2}\times$ base $\times$ height" is well known, in arbitrary triangles it can be nontrivial to compute the height. Calculus makes the problem simple.

Example 8.1.4 Finding the area of a triangle
Compute the area of the regions bounded by the lines $y = x+1$, $y = -2x+7$ and $y = -\frac{1}{2}x + \frac{5}{2}$, as shown in Figure 8.1.6.

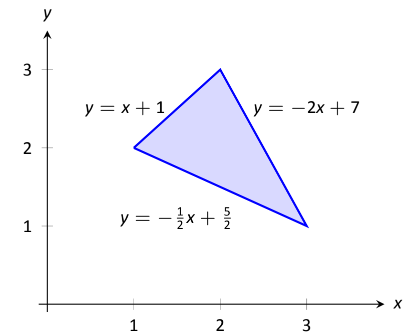


Figure 8.1.6: Graphing a triangular region in Example 8.1.4.

Solution Recognize that there are two "top" functions to this region, causing us to use two definite integrals.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big((x+1) - (-\frac{1}{2}x + \frac{5}{2})\big)\,dx + \int_2^3 \big((-2x+7) - (-\frac{1}{2}x + \frac{5}{2})\big)\,dx\\ &= 3/4 + 3/4\\ &= 3/2.\end{aligned}$$

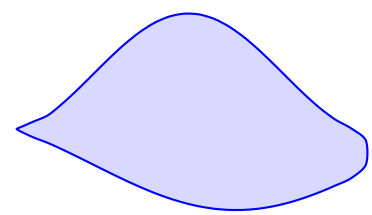
We can also approach this by converting each function into a function of y. This also requires 2 integrals, so there isn't really any advantage to doing so. We do it here for demonstration purposes.

The "top" function is always $x = \frac{7-y}{2}$ while there are two "bottom" functions. Being mindful of the proper integration bounds, we have

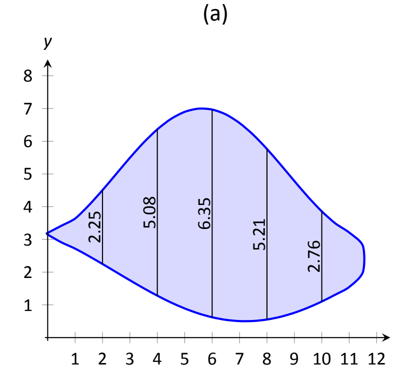
$$\begin{aligned}\text{Total Area} &= \int_1^2 \big(\frac{7-y}{2} - (5-2y)\big)\,dy + \int_2^3 \big(\frac{7-y}{2} - (y-1)\big)\,dy\\ &= 3/4 + 3/4\\ &= 3/2.\end{aligned}$$

Of course, the final answer is the same. (It is interesting to note that the area of all 4 subregions used is 3/4. This is coincidental.)

While we have focused on producing exact answers, we are also able to make approximations using the principle of Theorem 8.1.1. The integrand in the theo-rem is a distance ("top minus bottom"); integrating this distance function gives an area. By taking discrete measurements of distance, we can approximate an area using numerical integration techniques developed in Section 5.5. The fol-lowing example demonstrates this.



(a)



(b)

Figure 8.1.7: (a) A sketch of a lake, and (b) the lake with length measurements.

Example 8.1.5 Numerically approximating area
To approximate the area of a lake, shown in Figure 8.1.7 (a), the "length" of the lake is measured at 200-foot increments as shown in Figure 8.1.7 (b), where the lengths are given in hundreds of feet. Approximate the area of the lake.

Solution The measurements of length can be viewed as measuring "top minus bottom" of two functions. The exact answer is found by integrating $\int_0^{12} \big(f(x) - g(x)\big)\,dx$, but of course we don't know the functions f and g. Our discrete measurements instead allow us to approximate.

We have the following data points:

$$(0,0),\ (2,2.25),\ (4,5.08),\ (6,6.35),\ (8,5.21),\ (10,2.76),\ (12,0).$$

We also have that $\Delta x = \frac{b-a}{n} = 2$, so Simpson's Rule gives

$$\begin{aligned}\text{Area} &\doteq \frac{2}{3}\Big(1\cdot 0 + 4\cdot 2.25 + 2\cdot 5.08 + 4\cdot 6.35 + 2\cdot 5.21 + 4\cdot 2.76 + 1\cdot 0\Big)\\ &= 44.01\overline{3}\ \text{units}^2.\end{aligned}$$

Since the measurements are in hundreds of feet, $\text{units}^2 = (100\ \text{ft})^2 = 10,000\ \text{ft}^2$, giving a total area of $440,133\ \text{ft}^2$. (Since we are approximating, we'd likely say the area was about $440,000\ \text{ft}^2$, which is a little more than 10 acres.)

In the next section we apply our applications–of–integration techniques to finding the volumes of certain solids.

Use the *Five Step Procedure* in each problem

Exercises 8.1

Terms and Concepts

1. T/F: The area between curves is always positive.

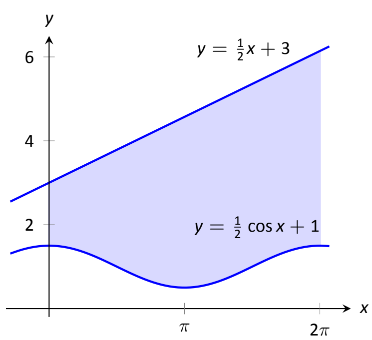
2. T/F: Calculus can be used to find the area of basic geometric shapes.

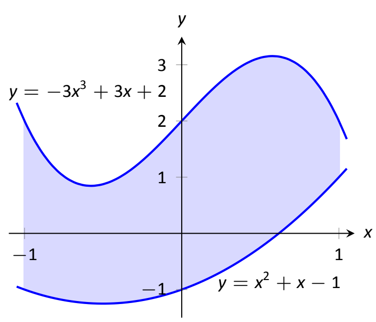
3. In your own words, describe how to find the total area enclosed by $y = f(x)$ and $y = g(x)$.

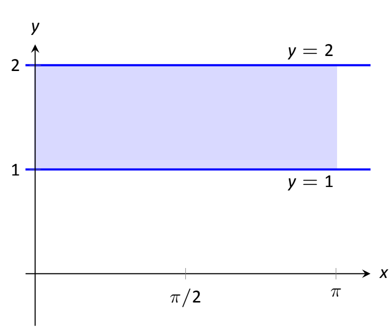
4. Describe a situation where it is advantageous to find an area enclosed by curves through integration with respect to y instead of x.

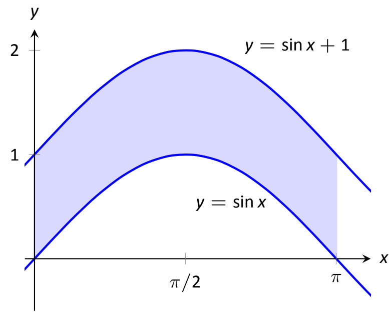
Problems

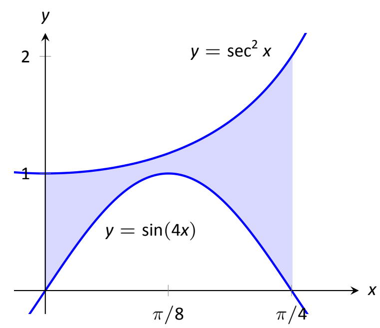
In Exercises 5 – 12, find the area of the shaded region in the given graph.

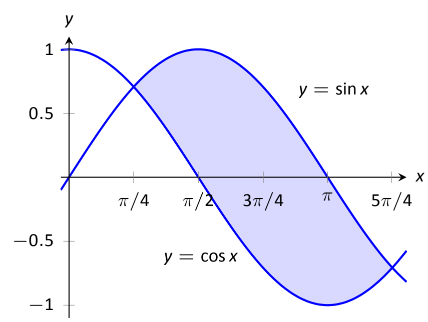
5. 

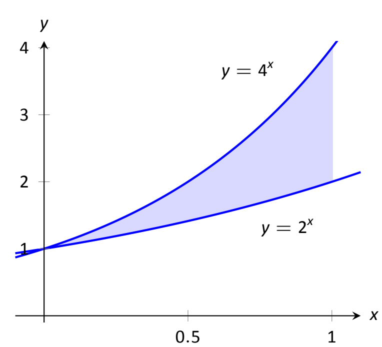
6. 

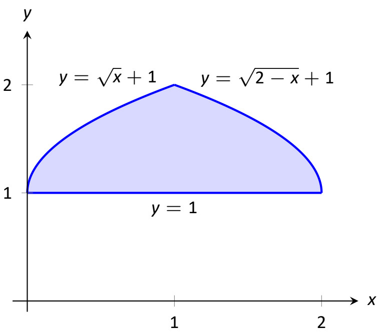
7. 

8. 

9. 

10. 

11. 

12. 

In Exercises 13 – 20, find the total area enclosed by the functions f and g.

13. $f(x) = 2x^2 + 5x - 3$, $g(x) = x^2 + 4x - 1$

14. $f(x) = x^2 - 3x + 2$, $g(x) = -3x + 3$

15. $f(x) = \sin x$, $g(x) = 2x/\pi$

16. $f(x) = x^3 - 4x^2 + x - 1$, $g(x) = -x^2 + 2x - 4$

17. $f(x) = x$, $g(x) = \sqrt{x}$

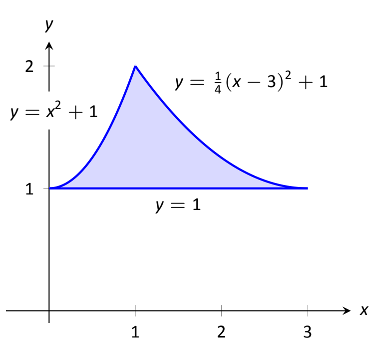
18. $f(x) = -x^3 + 5x^2 + 2x + 1$, $g(x) = 3x^2 + x + 3$

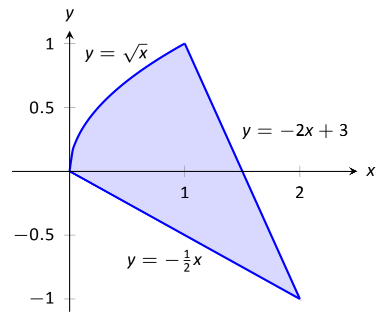
19. The functions $f(x) = \cos(x)$ and $g(x) = \sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

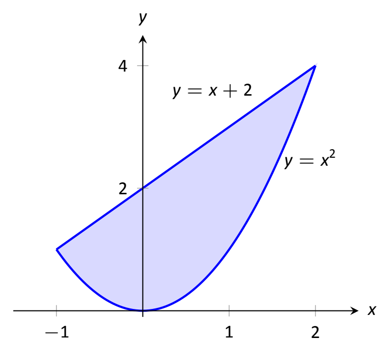
20. The functions $f(x) = \cos(2x)$ and $g(x) = \sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

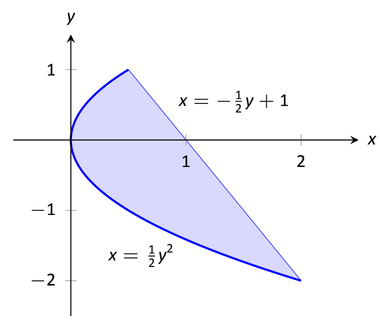
In Exercises 21 – 26, find the area of the enclosed region in two ways:

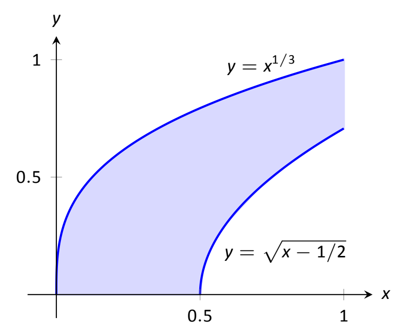
1. **by treating the boundaries as functions of x, and**
2. **by treating the boundaries as functions of y.**

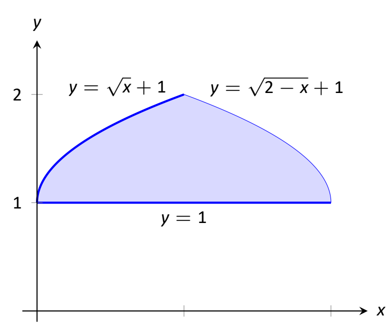
21. 

22. 

23. 

24. 

25. 

26. 

In Exercises 27 – 30, find the area triangle formed by the given three points.

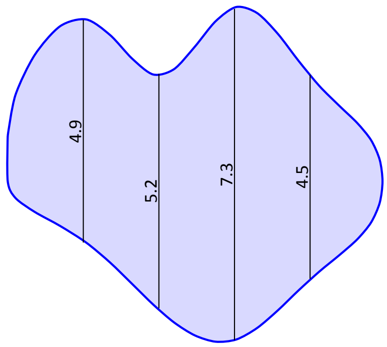
27. $(1,1)$, $(2,3)$, and $(3,3)$

28. $(-1,1)$, $(1,3)$, and $(2,-1)$

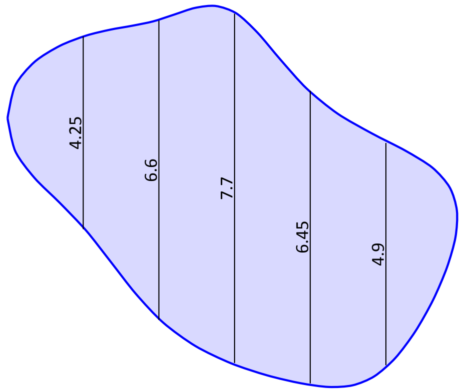
29. $(1,1)$, $(3,3)$, and $(3,3)$

30. $(0,0)$, $(2,5)$, and $(5,2)$

31. Use the Trapezoidal Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 100-foot increments.



32. Use Simpson's Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 200-foot increments.



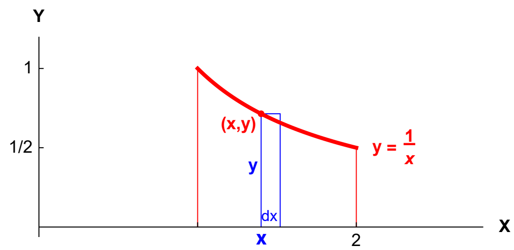
Solutions 8.1

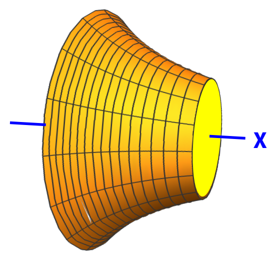
1. T
2. T
3. Answers will vary.
4. Answers may vary; one common answer is when the region has two or more "top" or "bottom" functions when viewing the region with respect to *x*, but has only 1 "top" function and 1 "bottom" function when viewed with respect to *y*. The former area requires multiple integrals to compute, whereas the latter area requires one.
5. $4\pi+\pi^2 \doteq 22.436$
6. $16/3$
7. π
8. π
9. $1/2$
10. $2\sqrt{2}$
11. $1/\ln 4$
12. $4/3$
13. 4.5
14. $4/3$
15. $2-\pi/2$
16. 8
17. $1/6$
18. $37/12$
19. All enclosed regions have the same area, with regions being the reflection of adjacent regions. One region is formed on $[\pi/4, 5\pi/4]$, with area $2\sqrt{2}$.
20. On regions such as $[\pi/6, 5\pi/6]$, the area is $3\sqrt{3}/2$. On regions such as $[-\pi/2, \pi/6]$, the area is $3\sqrt{3}/4$.
21. 1
22. $5/3$
23. $9/2$
24. $9/4$
25. $1/12(9-2\sqrt{2}) \doteq 0.514$
26. $4/3$
27. 1
28. 5
29. 4
30. $133/20$
31. 219,000 ft^2
32. 623,333 ft^2

8.2 Volumes by Slicing. The Disk Method: 5 Steps

Example Find the volume when the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$ and $x = 2$ is revolved about the X-axis.

I, II





When the differential element is rotated about the x-axis, the result is asymptotically a disk whose volume is
$dV \approx \pi y^2 dx$.

III $\quad dV \doteq \pi y^2 dx = \pi \frac{1}{x^2} dx$

IV $\quad V = \pi \int_1^2 \frac{dx}{x^2}$

V $\quad = \pi \left[-\frac{1}{x} \right]_1^2$

$= \frac{\pi}{2}$

8.2 Readings Volume by Cross-Sectional Area; Disk and Washer Methods

The volume of a general right cylinder, as shown in Figure 8.2.1, is

$$\text{Area of the base} \times \text{height}.$$

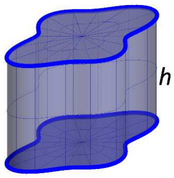
We can use this fact as the building block in finding volumes of a variety of shapes.

Given an arbitrary solid, we can *approximate* its volume by cutting it into n thin slices. When the slices are thin, each slice can be approximated well by a general right cylinder. Thus the volume of each slice is approximately its cross-sectional area $\times$ thickness. (These slices are the differential elements.)

By orienting a solid along the x-axis, we can let $A(x_i)$ represent the cross-sectional area of the i^{th} slice, and let Δx_i represent the thickness of this slice (the thickness is a small change in x). The total volume of the solid is approximately:

$$\begin{aligned}\text{Volume} &\doteq \sum_{i=1}^{n} \Big[\text{Area} \times \text{thickness}\Big]\\ &= \sum_{i=1}^{n} A(x_i)\Delta x_i.\end{aligned}$$

Recognize that this is a Riemann Sum. By taking a limit (as the thickness of the slices goes to 0) we can find the volume exactly.



base area $= A$
Volume $= A \cdot h$

Figure 8.2.1: The volume of a general right cylinder

Theorem 8.2.1 Volume By Cross-Sectional Area

The volume V of a solid, oriented along the x-axis with cross-sectional area $A(x)$ from $x = a$ to $x = b$, is

$$V = \int_a^b A(x)\,dx.$$

Example 8.2.1 Finding the volume of a solid

Find the volume of a pyramid with a square base of side length 10 in and a height of 5 in.

Solution There are many ways to "orient" the pyramid along the x-axis; Figure 8.2.2 gives one such way, with the pointed top of the pyramid at the origin and the x-axis going through the center of the base.

Each cross section of the pyramid is a square; this is a sample differential element. To determine its area $A(x)$, we need to determine the side lengths of

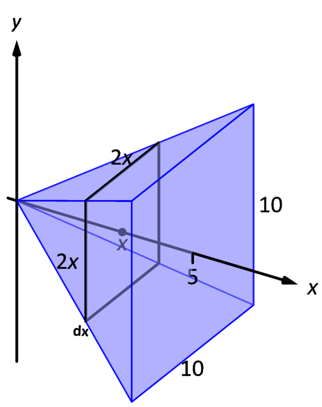


Figure 8.2.2: Orienting a pyramid along the x-axis in Example 8.2.1

$$\begin{aligned}dV &\approx A(x)\,dx\\ &= (2x)^2\,dx = 4x^2\,dx\end{aligned}$$

$$\begin{aligned}V &= 4\int_0^5 x^2 dx\\ &= 4\frac{x^3}{3}\Big|_0^5\\ &= \frac{500}{3}\end{aligned}$$

the square.

When $x = 5$, the square has side length 10; when $x = 0$, the square has side length 0. Since the edges of the pyramid are lines, it is easy to figure that each cross-sectional square has side length $2x$, giving $A(x) = (2x)^2 = 4x^2$.

If one were to cut a slice out of the pyramid at $x = 3$, as shown in Figure 8.2.3, one would have a shape with square bottom and top with sloped sides. If the slice were thin, both the bottom and top squares would have sides lengths of about 6, and thus the cross–sectional area of the bottom and top would be about 36in^2. Letting Δx_i represent the thickness of the slice, the volume of this slice would then be about $36\Delta x_i$in^3.

Cutting the pyramid into n slices divides the total volume into n equally–spaced smaller pieces, each with volume $(2x_i)^2\Delta x$, where x_i is the approximate location of the slice along the x-axis and Δx represents the thickness of each slice. One can approximate total volume of the pyramid by summing up the volumes of these slices:

$$\text{Approximate volume } = \sum_{i=1}^{n}(2x_i)^2\Delta x.$$

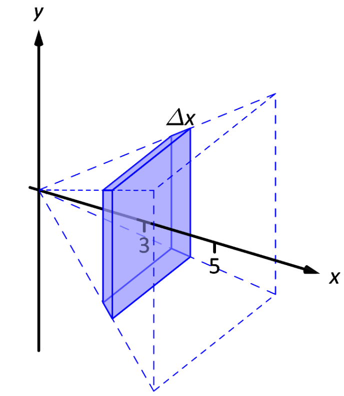


Figure 8.2.3: Cutting a slice in the pyramid in Example 8.2.1 at $x = 3$.

Taking the limit as $n \to \infty$ gives the actual volume of the pyramid; recoginizing this sum as a Riemann Sum allows us to find the exact answer using a definite integral, matching the definite integral given by Theorem 8.2.1.

We have

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n}(2x_i)^2\Delta x \\ &= \int_0^5 4x^2\,dx \\ &= \frac{4}{3}x^3\Big|_0^5 \\ &= \frac{500}{3}\text{ in}^3 \doteq 166.67\text{ in}^3. \end{aligned}$$

We can check our work by consulting the general equation for the volume of a pyramid (see the back cover under “Volume of A General Cone”):

$$\tfrac{1}{3} \times \text{area of base} \times \text{height}.$$

Certainly, using this formula from geometry is faster than our new method, but the calculus–based method can be applied to much more than just cones.

An important special case of Theorem 8.2.1 is when the solid is a **solid of revolution**, that is, when the solid is formed by rotating a shape around an axis.

Start with a function $y = f(x)$ from $x = a$ to $x = b$. Revolving this curve about a horizontal axis creates a three-dimensional solid whose cross sections are disks (thin circles). Let $R(x)$ represent the radius of the cross-sectional disk at x; the area of this disk is $\pi R(x)^2$. Applying Theorem 8.2.1 gives the Disk Method.

> **Key Idea 7.8.1 The Disk Method**
>
> Let a solid be formed by revolving the curve $y = f(x)$ from $x = a$ to $x = b$ around a horizontal axis, and let $R(x)$ be the radius of the cross-sectional disk at x. The volume of the solid is
>
> $$V = \pi \int_a^b R(x)^2\, dx.$$

Example 8.2.2 Finding volume using the Disk Method
Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, around the x-axis.

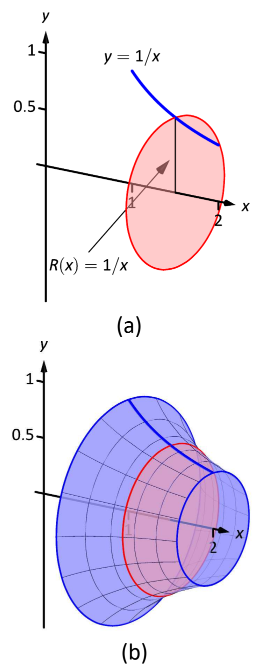


Figure 8.2.4: Sketching a solid in Example 8.2.2.

Solution A sketch can help us understand this problem. In Figure 8.2.4(a) the curve $y = 1/x$ is sketched along with the differential element – a disk – at x with radius $R(x) = 1/x$. In Figure 8.2.4 (b) the whole solid is pictured, along with the differential element.

The volume of the differential element shown in part (a) of the figure is approximately $\pi R(x_i)^2 \Delta x$, where $R(x_i)$ is the radius of the disk shown and Δx is the thickness of that slice. The radius $R(x_i)$ is the distance from the x-axis to the curve, hence $R(x_i) = 1/x_i$.

Slicing the solid into n equally–spaced slices, we can approximate the total volume by adding up the approximate volume of each slice:

$$\text{Approximate volume } = \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x.$$

Taking the limit of the above sum as $n \to \infty$ gives the actual volume; recognizing this sum as a Riemann sum allows us to evaluate the limit with a definite integral, which matches the formula given in Key Idea 8.2.1:

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x \\ &= \pi \int_1^2 \left(\frac{1}{x}\right)^2 dx \\ &= \pi \int_1^2 \frac{1}{x^2}\, dx \end{aligned}$$

$$= \pi \left[-\frac{1}{x} \right] \Big|_1^2$$
$$= \pi \left[-\frac{1}{2} - (-1) \right]$$
$$= \frac{\pi}{2} \text{ units}^3.$$

While Key Idea 8.2.1 is given in terms of functions of x, the principle involved can be applied to functions of y when the axis of rotation is vertical, not horizontal. We demonstrate this in the next example.

Example 7.2.3 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, about the y-axis.

Solution Since the axis of rotation is vertical, we need to convert the function into a function of y and convert the x-bounds to y-bounds. Since $y = 1/x$ defines the curve, we rewrite it as $x = 1/y$. The bound $x = 1$ corresponds to the y-bound $y = 1$, and the bound $x = 2$ corresponds to the y-bound $y = 1/2$.

Thus we are rotating the curve $x = 1/y$, from $y = 1/2$ to $y = 1$ about the y-axis to form a solid. The curve and sample differential element are sketched in Figure 8.2.5 (a), with a full sketch of the solid in Figure 8.2.5 (b). We integrate to find the volume:

$$V = \pi \int_{1/2}^{1} \frac{1}{y^2}\, dy$$
$$= -\frac{\pi}{y}\Big|_{1/2}^{1}$$
$$= \pi \text{ units}^3.$$

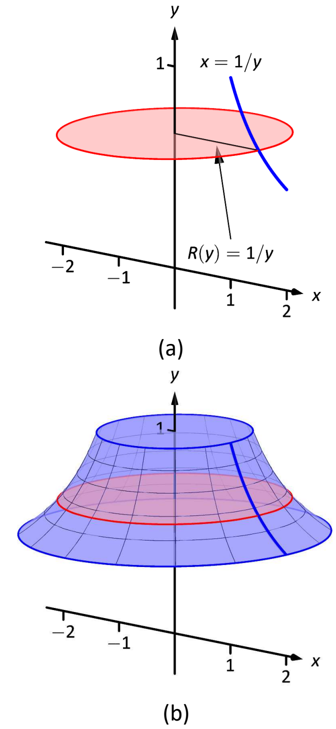


Figure 8.2.5: Sketching a solid in Example 8.2.3.

We can also compute the volume of solids of revolution that have a hole in the center. The general principle is simple: compute the volume of the solid irrespective of the hole, then subtract the volume of the hole. If the outside radius of the solid is $R(x)$ and the inside radius (defining the hole) is $r(x)$, then the volume is

$$V = \pi \int_a^b R(x)^2\, dx - \pi \int_a^b r(x)^2\, dx = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right)\, dx.$$

One can generate a solid of revolution with a hole in the middle by revolving a region about an axis. Consider Figure 8.2.6(a), where a region is sketched along

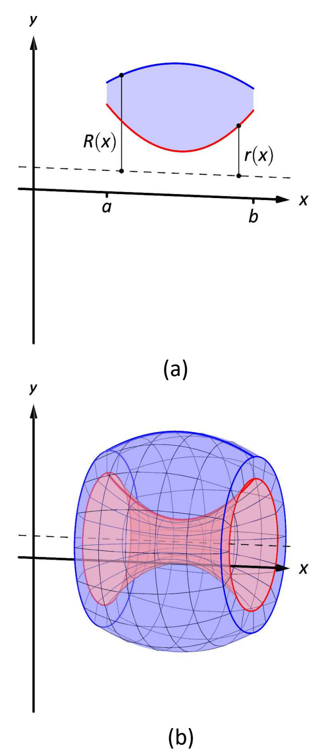


Figure 8.2.6: Establishing the Washer Method; see also Figure 8.2.7.

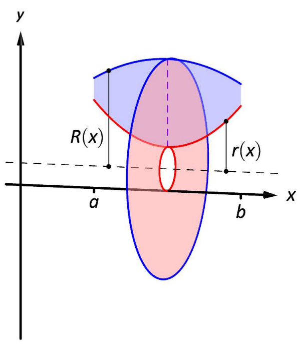


Figure 8.2.7: Establishing the Washer Method; see also Figure 8.2.6.

with a dashed, horizontal axis of rotation. By rotating the region about the axis, a solid is formed as sketched in Figure 8.2.6(b). The outside of the solid has radius $R(x)$, whereas the inside has radius $r(x)$. Each cross section of this solid will be a washer (a disk with a hole in the center) as sketched in Figure 8.2.7. This leads us to the Washer Method.

> **Key Idea 7.2.2 The Washer Method**
>
> Let a region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ be rotated about a horizontal axis that does not intersect the region, forming a solid. Each cross section at x will be a washer with outside radius $R(x)$ and inside radius $r(x)$. The volume of the solid is
>
> $$V = \pi \int_a^b \Big(R(x)^2 - r(x)^2\Big)\,dx.$$

Even though we introduced it first, the Disk Method is just a special case of the Washer Method with an inside radius of $r(x) = 0$.

Example 7.2.4 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the x-axis.

Solution A sketch of the region will help, as given in Figure 8.2.8(a). Rotating about the x-axis will produce cross sections in the shape of washers, as shown in Figure 8.2.8(b); the complete solid is shown in part (c). The outside radius of this washer is $R(x) = 2x + 1$; the inside radius is $r(x) = x^2 - 2x + 2$. As the region is bounded from $x = 1$ to $x = 3$, we integrate as follows to compute the volume.

$$\begin{aligned} V &= \pi \int_1^3 \Big((2x-1)^2 - (x^2-2x+2)^2\Big)\,dx \\ &= \pi \int_1^3 \Big(-x^4 + 4x^3 - 4x^2 + 4x - 3\Big)\,dx \\ &= \pi \Big[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 + 2x^2 - 3x\Big]\Big|_1^3 \\ &= \frac{104}{\text{Mp}} \end{aligned}$$

When rotating about a vertical axis, the outside and inside radius functions must be functions of y.

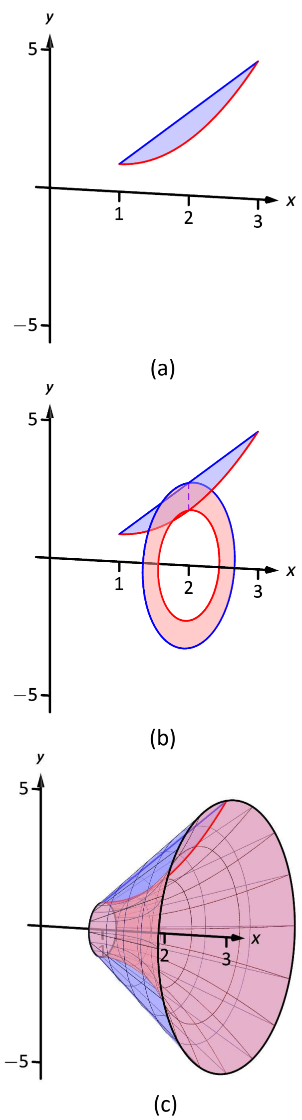


Figure 8.2.8: Sketching the differential el-ement and solid in Example 8.2.4.

Example 8.2.5 Finding volume with the Washer Method
Find the volume of the solid formed by rotating the triangular region with vertices at $(1, 1)$, $(2, 1)$ and $(2, 3)$ about the y-axis.

Solution The triangular region is sketched in Figure 8.2.9(a); the differential element is sketched in (b) and the full solid is drawn in (c). They help us establish the outside and inside radii. Since the axis of rotation is vertical, each radius is a function of y.

The outside radius $R(y)$ is formed by the line connecting $(2, 1)$ and $(2, 3)$; it is a constant function, as regardless of the y-value the distance from the line to the axis of rotation is 2. Thus $R(y) = 2$.

The inside radius is formed by the line connecting $(1, 1)$ and $(2, 3)$. The equation of this line is $y = 2x - 1$, but we need to refer to it as a function of y. Solving for x gives $r(y) = \frac{1}{2}(y + 1)$.

We integrate over the y-bounds of $y = 1$ to $y = 3$. Thus the volume is

$$\begin{aligned} V &= \pi \int_1^3 \left(2^2 - \left(\frac{1}{2}(y+1)\right)^2\right) dy \\ &= \pi \int_1^3 \left(-\frac{1}{4}y^2 - \frac{1}{2}y + \frac{15}{4}\right) dy \\ &= \pi \left[-\frac{1}{12}y^3 - \frac{1}{4}y^2 + \frac{15}{4}y\right]\Big|_1^3 \\ &= \frac{10}{3}\pi \doteq 10.47 \text{ units}^3. \end{aligned}$$

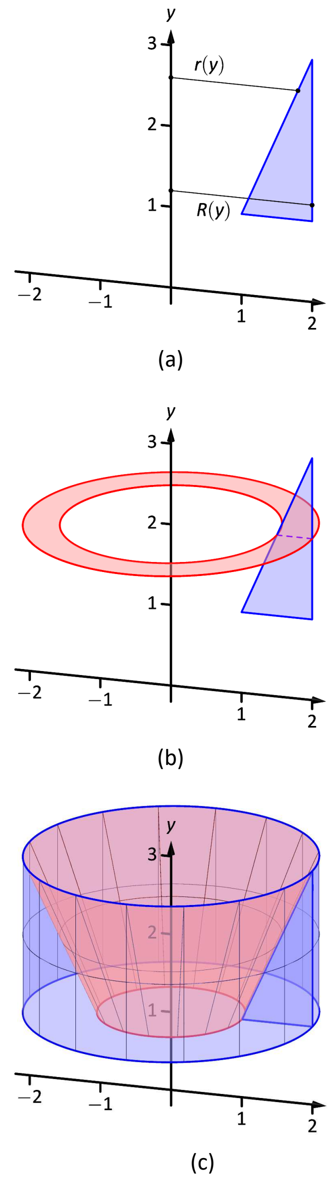


Figure 8.2.9: Sketching the solid in Example 8.2.5.

This section introduced a new application of the definite integral. Our default view of the definite integral is that it gives "the area under the curve." However, we can establish definite integrals that represent other quantities; in this section, we computed volume.

The ultimate goal of this section is not to compute volumes of solids. That can be useful, but what is more useful is the understanding of this basic principle of integral calculus, outlined in Key Idea 8.0.1: to find the exact value of some quantity,

- we start with an approximation (in this section, slice the solid and approximate the volume of each slice),
- then make the approximation better by refining our original approximation (i.e., use more slices),
- then use limits to establish a definite integral which gives the exact value.

We practice this principle in the next section where we find volumes by slicing solids in a different way.

Use the *Five Step Procedure* in each problem

Exercises 8.2

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

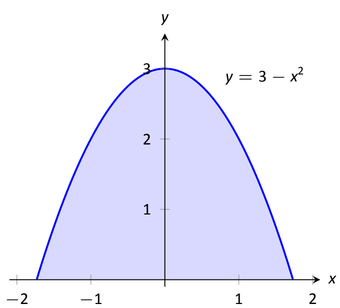
2. In your own words, explain how the Disk and Washer Methods are related.

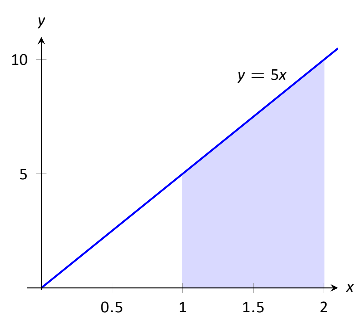
3. Explain the how the units of volume are found in the integral of Theorem 7.2.1: if $A(x)$ has units of in^2, how does $\int A(x)\,dx$ have units of in^3?

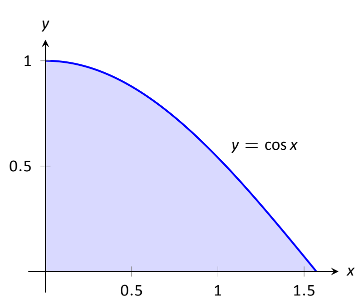
4. A fundamental principle of this section is "_________ can be found by integrating an area function."

Problems

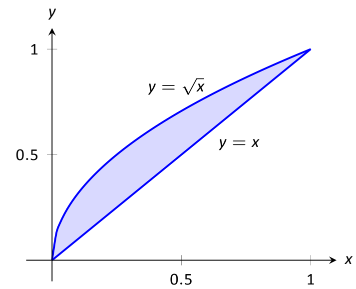
In Exercises 5 – 8, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the x-axis.

5. 

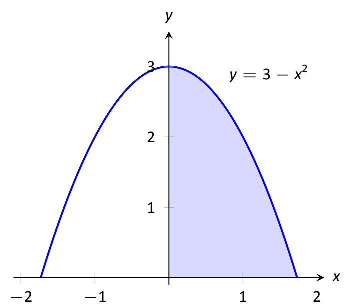
6. 

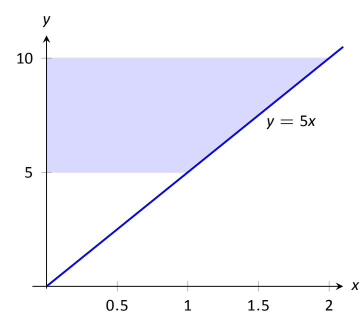
7. 

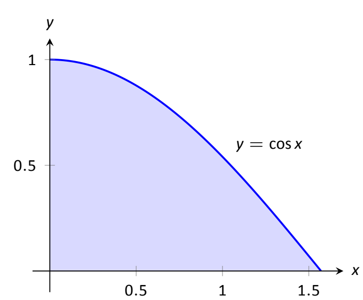
It is an excellent exercise to translate each of the examples of Apex above into the 5 step Method.

8. 

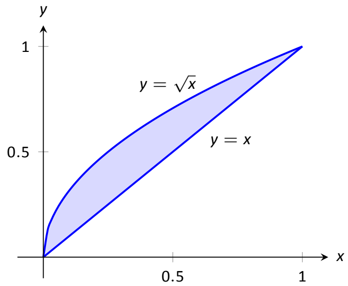
In Exercises 9 – 12, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the y-axis.

9. 

10. 

11. 

(Hint: Integration By Parts will be necessary, twice. First let $u = \arccos^2 x$, then let $u = \arccos x$.)

12. 

In Exercises 13 – 18, a region of the Cartesian plane is described. Use the Disk/Washer Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:
    (a) the x-axis
    (b) $y = 1$
    (c) the y-axis
    (d) $x = 1$

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:
    (a) the x-axis
    (b) $y = 4$
    (c) $y = -1$
    (d) $x = 2$

15. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.
Rotate about:
    (a) the x-axis
    (b) $y = 2$
    (c) the y-axis
    (d) $x = 1$

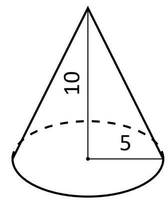
16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:
    (a) the x-axis
    (b) $y = 1$
    (c) $y = 5$

17. Region bounded by $y = 1/\sqrt{x^2 + 1}$, $x = -1$, $x = 1$ and the x-axis.
Rotate about:
    (a) the x-axis
    (b) $y = 1$
    (c) $y = -1$

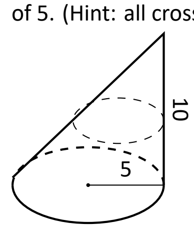
18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:
    (a) the x-axis
    (b) $y = 4$
    (c) the y-axis
    (d) $x = 2$

In Exercises 19 – 22, a solid is described. Orient the solid along the x-axis such that a cross-sectional area function $A(x)$ can be obtained, then apply Theorem 7.2.1 to find the volume of the solid.

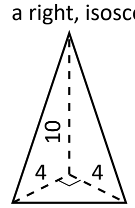
19. A right circular cone with height of 10 and base radius of 5.



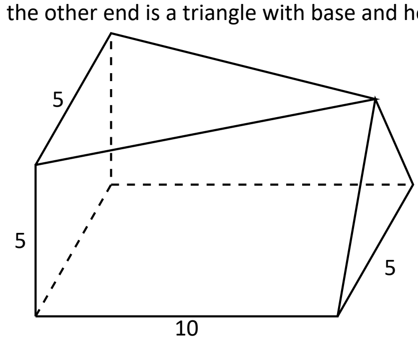
20. A skew right circular cone with height of 10 and base radius of 5. (Hint: all cross-sections are circles.)



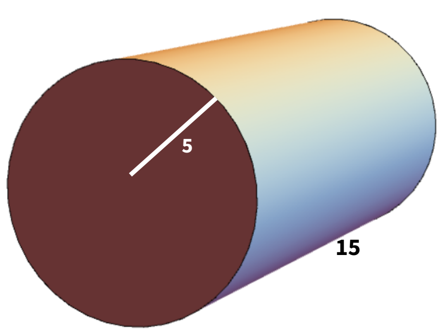
21. A right triangular cone with height of 10 and whose base is a right, isosceles triangle with side length 4.

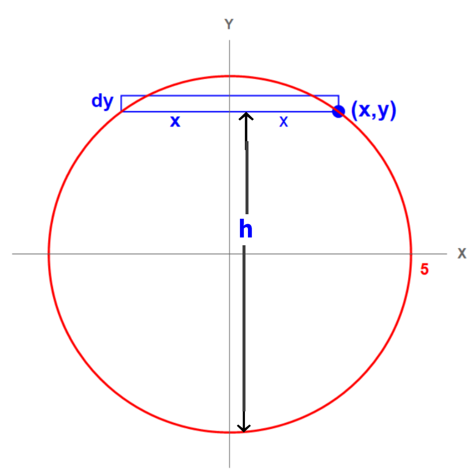


22. A solid with length 10 with a rectangular base and triangular top, wherein one end is a square with side length 5 and the other end is a triangle with base and height of 5.



23. Find the volume of water in the tank if it is filled to height h.





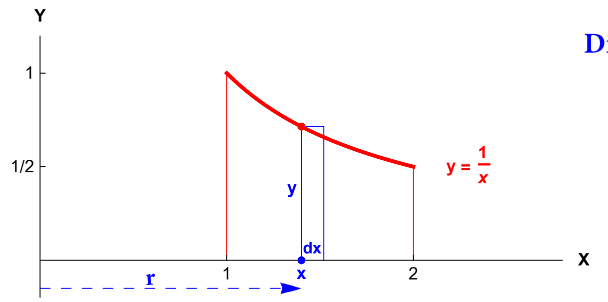
Section 8.2

1. T
2. Answers will vary.
3. Recall that "dx" does not just "sit there;" it is multiplied by $A(x)$ and represents the thickness of a small slice of the solid. Therefore dx has units of in, giving $A(x)\ dx$ the units of in^3.
4. volume
5. $48\pi\sqrt{3}/5$ units^3
6. $175\pi/3$ units^3
7. $\pi^2/4$ units^3
8. $\pi/6$ units^3
9. $9\pi/2$ units^3
10. $35\pi/3$ units^3
11. $\pi^2 - 2\pi$ units^3
12. $2\pi/15$ units^3
13. (a) $\pi/2$
    (b) $5\pi/6$
    (c) $4\pi/5$
    (d) $8\pi/15$
14. (a) $512\pi/15$
    (b) $256\pi/5$
    (c) $832\pi/15$
    (d) $128\pi/3$
15. (a) $4\pi/3$
    (b) $2\pi/3$
    (c) $4\pi/3$
    (d) $\pi/3$
16. (a) $104\pi/15$
    (b) $64\pi/15$
    (c) $32\pi/5$
17. (a) $\pi^2/2$
    (b) $\pi^2/2 - 4\pi\sinh^{-1}(1)$
    (c) $\pi^2/2 + 4\pi\sinh^{-1}(1)$
18. (a) 8π
    (b) 8π
    (c) $16\pi/3$
    (d) $8\pi/3$
19. Placing the tip of the cone at the origin such that the x-axis runs through the center of the circular base, we have $A(x) = \pi x^2/4$. Thus the volume is $250\pi/3$ units^3.
20. The cross–sections of this cone are the same as the cone in Exercise 19. Thus they have the same volume of $250\pi/3$ units^3.
21. Orient the cone such that the tip is at the origin and the x-axis is perpendicular to the base. The cross–sections of this cone are right, isosceles triangles with side length $2x/5$; thus the cross–sectional areas are $A(x) = 2x^2/25$, giving a volume of $80/3$ units^3.
22. Orient the solid so that the x-axis is parallel to long side of the base. All cross–sections are trapezoids (at the far left, the trapezoid is a square; at the far right, the trapezoid has a top length of 0, making it a triangle). The area of the trapezoid at x is $A(x) = 1/2(-1/2x + 5 + 5)(5) = -5/4x + 25$. The volume is 187.5 units^3.
23. $\mathbf{V} = 15\,(h-5)\sqrt{h(10-h)} + 375\sin^{-1}\left(\dfrac{h-5}{5}\right) + \dfrac{375\,\pi}{2}$. **If you can work this one, you should get an A$^+$.**

8.3 Volumes by Cylindrical Shell Method: 5 Steps

Find the volume when the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$ and $x = 2$ is revolved about the Y-axis.

I, II.



Distance around:

$c = 2\pi x$

III. $dV \approx 2\pi x\, y\, dx \;=\; 2\pi x \cdot \frac{1}{x} dx \;=\; 2\pi dx$

IV. $V = 2\pi \int_1^2 dx$

V. $= 2\pi x\big]_1^2 \;=\; 2\pi$

Note that dV is independent of dx.
Can you explain this intuitively?

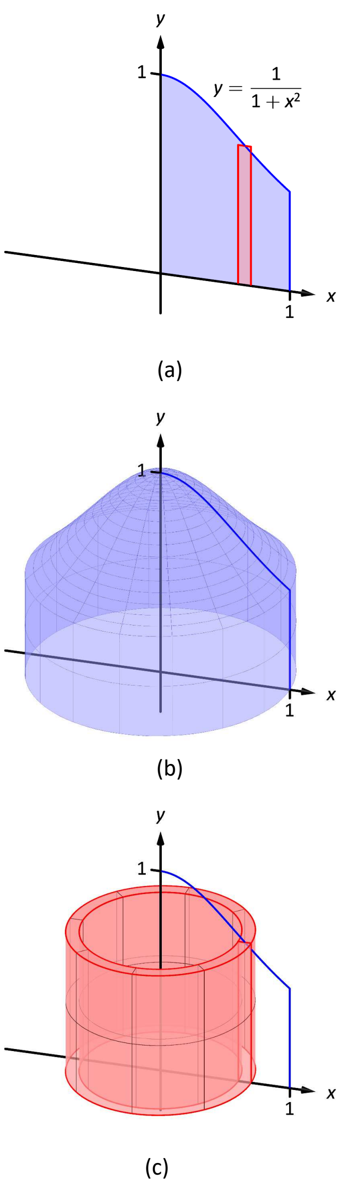


Figure 8.3.1: Introducing the Shell Method.

8.3 Readings The Shell Method

Often a given problem can be solved in more than one way. A particular method may be chosen out of convenience, personal preference, or perhaps necessity. Ultimately, it is good to have options.

The previous section introduced the Disk and Washer Methods, which computed the volume of solids of revolution by integrating the cross–sectional area of the solid. This section develops another method of computing volume, the **Shell Method.** Instead of slicing the solid perpendicular to the axis of rotation creating cross-sections, we now slice it parallel to the axis of rotation, creating "shells."

Consider Figure 8.3.1, where the region shown in (a) is rotated around the *y*-axis forming the solid shown in (b). A small slice of the region is drawn in (a), parallel to the axis of rotation. When the region is rotated, this thin slice forms a **cylindrical shell**, as pictured in part (c) of the figure. The previous section approximated a solid with lots of thin disks (or washers); we now approximate a solid with many thin cylindrical shells.

To compute the volume of one shell, first consider the paper label on a soup can with radius r and height h. What is the area of this label? A simple way of determining this is to cut the label and lay it out flat, forming a rectangle with height h and length $2\pi r$. Thus the area is $A = 2\pi r\, h$; see Figure 8.3.2(a).

Do a similar process with a cylindrical shell, with height h, thickness Δx, and approximate radius r. Cutting the shell and laying it flat forms a rectangular solid with length $2\pi r$, height h and depth Δx. Thus the volume is $V \approx 2\pi rh\Delta x$; see Figure 8.3.2(b). (We say "approximately" since our radius was an approximation.)

By breaking the solid into n cylindrical shells, we can approximate the volume of the solid as

$$V \doteq \sum_{i=1}^{N} 2\pi r_i h_i \Delta x_i,$$

where r_i, h_i and Δx_i are the radius, height and thickness of the i^{th} shell, respectively.

This is a Riemann Sum. Rounding off yields the definite integral.

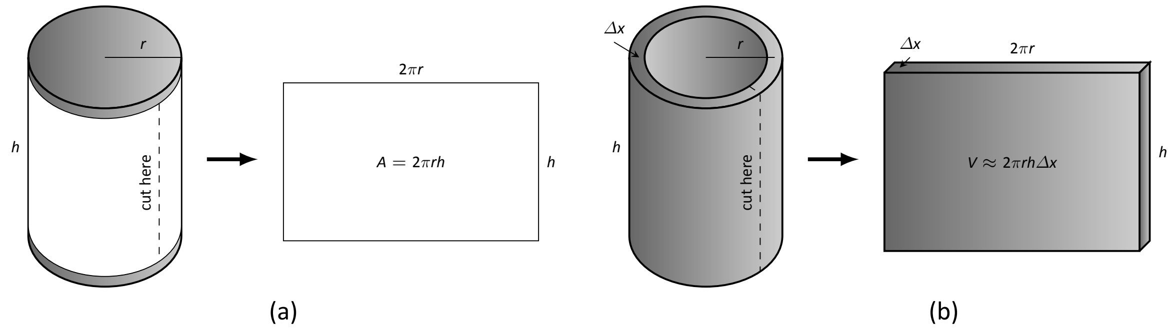


Figure 8.3.2: Determining the volume of a thin cylindrical shell.

Key Idea 8.3.1 The Shell Method

Let a solid be formed by revolving a region R, bounded by $x = a$ and $x = b$, around a vertical axis. Let $r(x)$ represent the distance from the axis of rotation to x (i.e., the radius of a sample shell) and let $h(x)$ represent the height of the solid at x (i.e., the height of the shell). The volume of the solid is

$$V = 2\pi \int_a^b r(x)h(x)\,dx.$$

Special Cases:

1. When the region R is bounded above by $y = f(x)$ and below by $y = g(x)$, then $h(x) = f(x) - g(x)$.

2. When the axis of rotation is the y-axis (i.e., $x = 0$) then $r(x) = x$.

Let's practice using the Shell Method.

Example 8.3.1 Finding volume using the Shell Method
Find the volume of the solid formed by rotating the region bounded by $y = 0$, $y = 1/(1 + x^2)$, $x = 0$ and $x = 1$ about the y-axis.

Solution This is the region used to introduce the Shell Method in Figure 8.3.1, but is sketched again in Figure 8.3.3 for closer reference. A line is drawn in the region parallel to the axis of rotation representing a shell that will

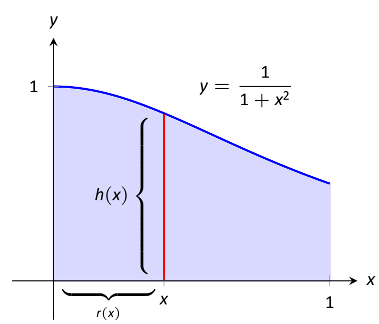


Figure 8.3.3: Graphing a region in Example 8.3.1.

be carved out as the region is rotated about the y-axis. (This is the differential element.)

The distance this line is from the axis of rotation determines $r(x)$; as the distance from x to the y-axis is x, we have $r(x) = x$. The height of this line determines $h(x)$; the top of the line is at $y = 1/(1+x^2)$, whereas the bottom of the line is at $y = 0$. Thus $h(x) = 1/(1+x^2) - 0 = 1/(1+x^2)$. The region is bounded from $x = 0$ to $x = 1$, so the volume is

$$V = 2\pi \int_0^1 \frac{x}{1+x^2}\,dx.$$

This requires substitution. Let $u = 1+x^2$, so $du = 2x\,dx$. We also change the bounds: $u(0) = 1$ and $u(1) = 2$. Thus we have:

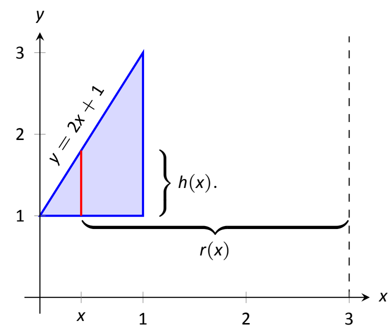
$$\begin{aligned} &= \pi \int_1^2 \frac{1}{u}\,du \\ &= \pi \ln u \Big|_1^2 \\ &= \pi \ln 2 \doteq 2.178 \text{ units}^3. \end{aligned}$$

Note: in order to find this volume using the Disk Method, two integrals would be needed to account for the regions above and below $y = 1/2$.

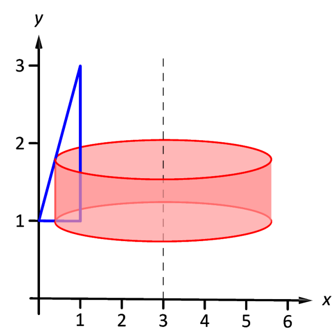
With the Shell Method, nothing special needs to be accounted for to compute the volume of a solid that has a hole in the middle, as demonstrated next.

Example 7.3.2 Finding volume using the Shell Method

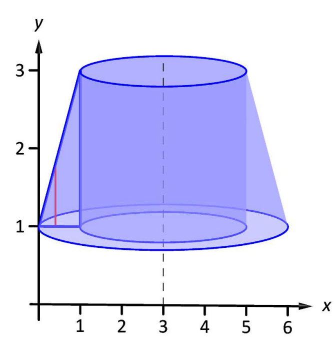
Find the volume of the solid formed by rotating the triangular region determined by the points $(0,1)$, $(1,1)$ and $(1,3)$ about the line $x = 3$.



(a)



(b)



(c)

Figure 8.3.4: Graphing a region in Example 8.3.2.

Solution The region is sketched in Figure 8.3.4(a) along with the differential element, a line within the region parallel to the axis of rotation. In part (b) of the figure, we see the shell traced out by the differential element, and in part (c) the whole solid is shown.

The height of the differential element is the distance from $y = 1$ to $y = 2x+1$, the line that connects the points $(0,1)$ and $(1,3)$. Thus $h(x) = 2x+1-1 = 2x$. The radius of the shell formed by the differential element is the distance from x to $x = 3$; that is, it is $r(x) = 3-x$. The x-bounds of the region are $x = 0$ to

$x = 1$, giving

$$\begin{aligned} V &= 2\pi \int_0^1 (3-x)(2x)\,dx \\ &= 2\pi \int_0^1 \left(6x - 2x^2\right)\,dx \\ &= 2\pi \left(3x^2 - \frac{2}{3}x^3\right)\Big|_0^1 \\ &= \frac{14}{3}\pi \doteq 14.66 \text{ units}^3. \end{aligned}$$

When revolving a region around a horizontal axis, we must consider the radius and height functions in terms of y, not x.

Example 8.3.3 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region given in Example 8.3.2 about the x-axis.

Solution The region is sketched in Figure 8.3.5(a) with a sample differential element. In part (b) of the figure the shell formed by the differential element is drawn, and the solid is sketched in (c). (Note that the triangular region looks "short and wide" here, whereas in the previous example the same region looked "tall and narrow." This is because the bounds on the graphs are different.)

The height of the differential element is an x-distance, between $x = \frac{1}{2}y - \frac{1}{2}$ and $x = 1$. Thus $h(y) = 1 - (\frac{1}{2}y - \frac{1}{2}) = -\frac{1}{2}y + \frac{3}{2}$. The radius is the distance from y to the x-axis, so $r(y) = y$. The y bounds of the region are $y = 1$ and $y = 3$, leading to the integral

$$\begin{aligned} V &= 2\pi \int_1^3 \left[y\left(-\frac{1}{2}y + \frac{3}{2}\right)\right]\,dy \\ &= 2\pi \int_1^3 \left[-\frac{1}{2}y^2 + \frac{3}{2}y\right]\,dy \\ &= 2\pi \left[-\frac{1}{6}y^3 + \frac{3}{4}y^2\right]\Big|_1^3 \\ &= 2\pi \left[\frac{9}{4} - \frac{7}{12}\right] \\ &= \frac{10}{3}\pi \doteq 10.472 \text{ units}^3. \end{aligned}$$

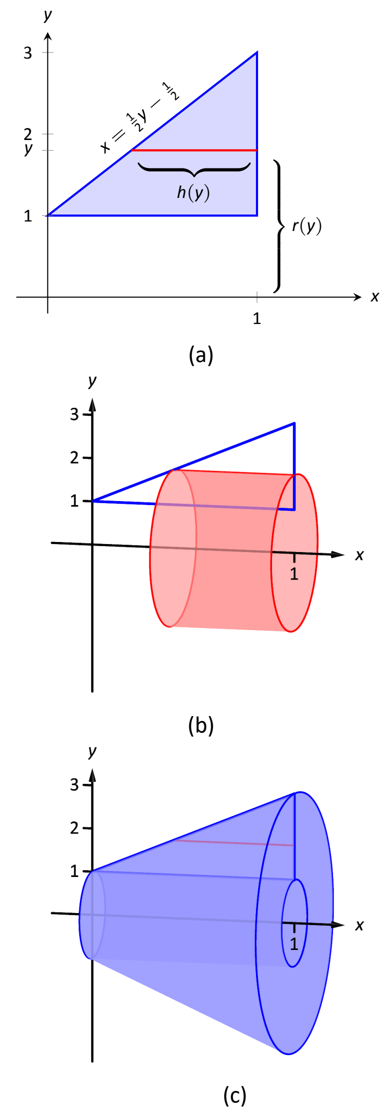


Figure 8.3.5: Graphing a region in Example 8.3.3.

At the beginning of this section it was stated that "it is good to have options." The next example finds the volume of a solid rather easily with the Shell Method, but using the Washer Method would be quite a chore.

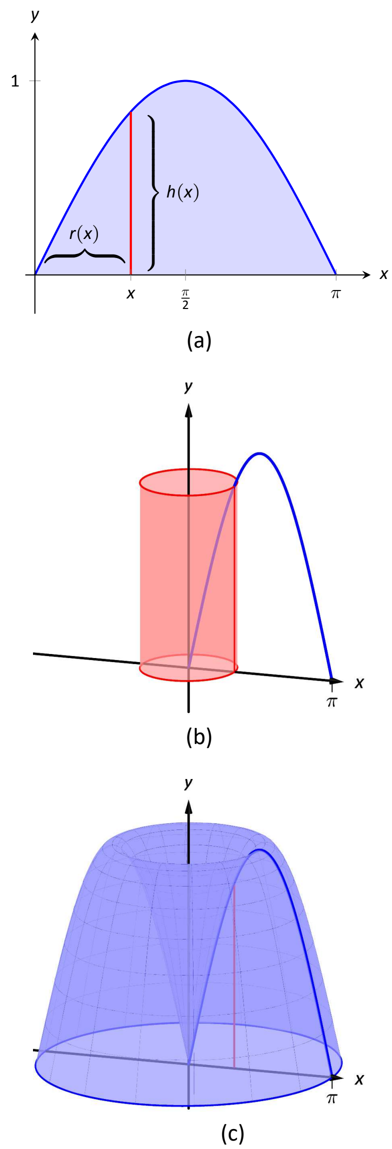


Figure 8.3.6: Graphing a region in Example 8.3.4.

Example 8.3.4 Finding volume using the Shell Method

Find the volume of the solid formed by revolving the region bounded by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ about the y-axis.

Solution The region and a differential element, the shell formed by this differential element, and the resulting solid are given in Figure 8.3.6. The radius of a sample shell is $r(x) = x$; the height of a sample shell is $h(x) = \sin x$, each from $x = 0$ to $x = \pi$. Thus the volume of the solid is

$$V = 2\pi \int_0^{\pi} x \sin x \, dx.$$

This requires Integration By Parts. Set $u = x$ and $dv = \sin x \, dx$; we leave it to the reader to fill in the rest. We have:

$$\begin{aligned}
&= 2\pi\left[-x\cos x\Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx\right] \\
&= 2\pi\left[\pi + \sin x\Big|_0^{\pi}\right] \\
&= 2\pi\left[\pi + 0\right] \\
&= 2\pi^2 \doteq 19.74 \text{ units}^3.
\end{aligned}$$

Note that in order to use the Washer Method, we would need to solve $y = \sin x$ for x, requiring the use of the arcsine function. We leave it to the reader to verify that the outside radius function is $R(y) = \pi - \arcsin y$ and the inside radius function is $r(y) = \arcsin y$. Thus the volume can be computed as

$$\pi \int_0^1 \left[(\pi - \arcsin y)^2 - (\arcsin y)^2\right] dy.$$

This integral isn't terrible given that the $\arcsin^2 y$ terms cancel, but it is more onerous than the integral created by the Shell Method.

We end this section with a table summarizing the usage of the Washer and Shell Methods.

Key Idea 8.3.2 Summary of the Washer and Shell Methods

Let a region R be given with x-bounds $x = a$ and $x = b$ and y-bounds $y = c$ and $y = d$.

	Washer Method	Shell Method
Horizontal Axis	$\pi \int_a^b \left(R(x)^2 - r(x)^2\right)\, dx$	$2\pi \int_c^d r(y)h(y)\, dy$
Vertical Axis	$\pi \int_c^d \left(R(y)^2 - r(y)^2\right)\, dy$	$2\pi \int_a^b r(x)h(x)\, dx$

As in the previous section, the real goal of this section is not to be able to compute volumes of certain solids. Rather, it is to be able to solve a problem by first approximating, then using limits to refine the approximation to give the exact value. In this section, we approximate the volume of a solid by cutting it into thin cylindrical shells. By summing up the volumes of each shell, we get an approximation of the volume. By taking a limit as the number of equally spaced shells goes to infinity, our summation can be evaluated as a definite integral, giving the exact value.

We use this same principle again in the next section, where we find the length of curves in the plane.

Use the *Five Step Procedure* in each problem

Exercises 8.3

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

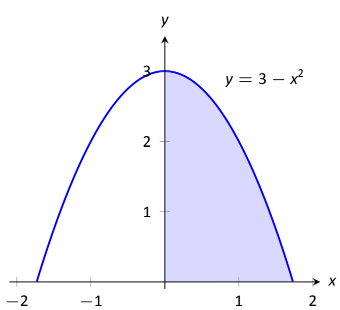
2. T/F: The Shell Method can only be used when the Washer Method fails.

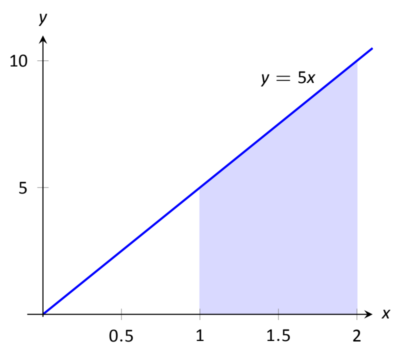
3. T/F: The Shell Method works by integrating cross–sectional areas of a solid.

4. T/F: When finding the volume of a solid of revolution that was revolved around a vertical axis, the Shell Method integrates with respect to x.

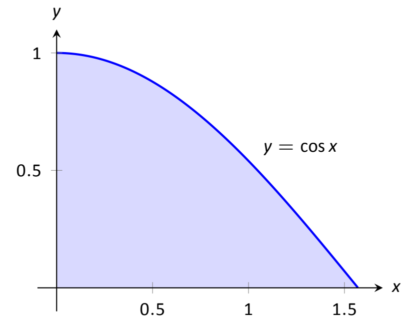
Problems

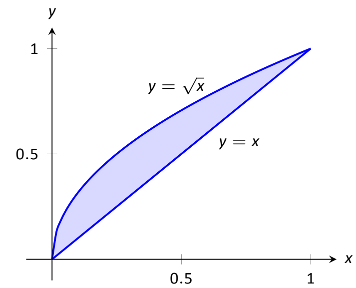
In Exercises 5 – 8, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the y-axis.

5. 

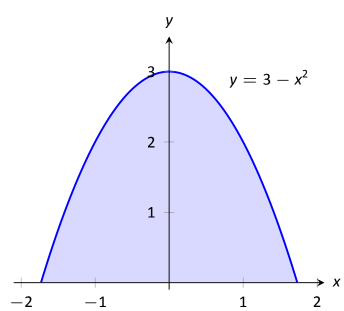
6. 

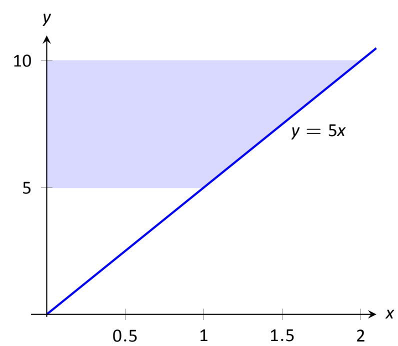
Again, it is an excellent exercise to translate each of the examples of Apex above into the 5 step Method.

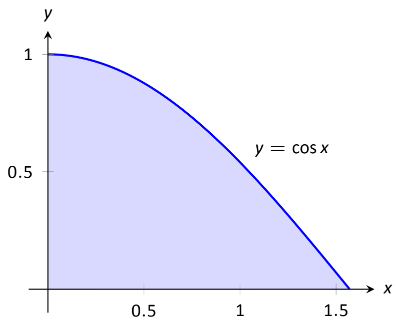
7. 

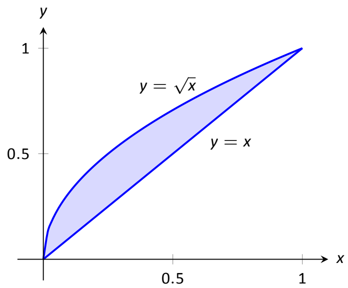
8. 

In Exercises 9 – 12, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the x-axis.

9. 

10. 

11. 

12. 

In Exercises 13 – 18, a region of the Cartesian plane is described. Use the Shell Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:
(a) the y-axis
(b) $x = 1$
(c) the x-axis
(d) $y = 1$

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:
(a) $x = 2$
(b) $x = -2$
(c) the x-axis
(d) $y = 4$

15. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.
Rotate about:
(a) the y-axis
(b) $x = 1$
(c) the x-axis
(d) $y = 2$

16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:
(a) the y-axis
(b) $x = 1$
(c) $x = -1$

17. Region bounded by $y = 1/\sqrt{x^2 + 1}$, $x = 1$ and the x and y-axes.
Rotate about:
(a) the y-axis
(b) $x = 1$

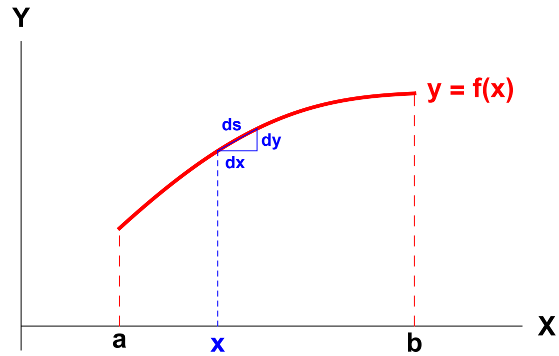
18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:
(a) the y-axis
(b) $x = 2$
(c) the x-axis
(d) $y = 4$

Solutions 8.3

1. T
2. F
3. F
4. T
5. $9\pi/2$ units^3
6. $70\pi/3$ units^3
7. $\pi^2 - 2\pi$ units^3
8. $2\pi/15$ units^3
9. $48\pi\sqrt{3}/5$ units^3
10. $350\pi/3$ units^3
11. $\pi^2/4$ units^3
12. $\pi/6$ units^3
13. (a) $4\pi/5$
(b) $8\pi/15$
(c) $\pi/2$
(d) $5\pi/6$
14. (a) $128\pi/3$
(b) $128\pi/3$
(c) $512\pi/15$
(d) $256\pi/5$
15. (a) $4\pi/3$
(b) $\pi/3$
(c) $4\pi/3$
(d) $2\pi/3$
16. (a) $16\pi/3$
(b) $8\pi/3$
(c) 8π
17. (a) $2\pi\left(\sqrt{2} - 1\right)$
(b) $2\pi(1 - \sqrt{2} + \sinh^{-1}(1))$
18. (a) $16\pi/3$
(b) $8\pi/3$
(c) 8π
(d) 8π

8.4 Arc Length

What is the ***arc length*, the** length of the curve, $y = f(x)$, $a \le x \le b$?



Theorem Let $y = f(x)$ be differentiable for $a \le x \le b$. Then its arc length on that interval is

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Derivation

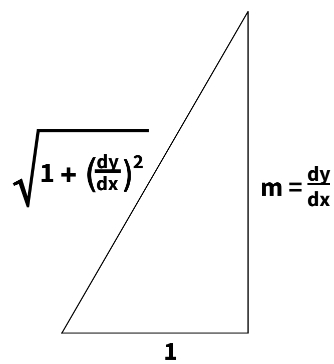
f is differentiable on the interval. So f is differentiable a x and therefore locally linear or equivalently 'asymptotically straight' there. Therefore

$$ds^2 \approx dx^2 + dy^2 \qquad \text{Theorem of Pythagoras}$$

$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right] dx^2$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \Longrightarrow \qquad s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

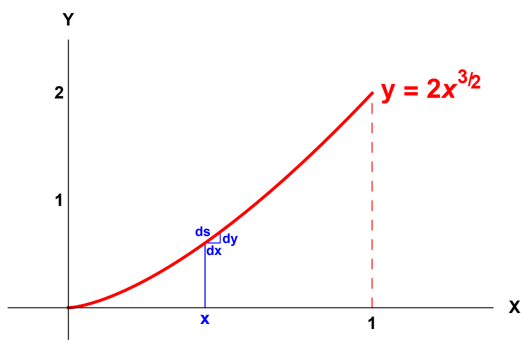
Note diagram below



meaning of slope

Example Find the length of $y = 2x^{3/2}$ for $0 \le x \le 1$.

I., II.



III. $y = 2x^{3/2} \Rightarrow \frac{dy}{dx} = 3x^{1/2}$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 9x}$$

IV. $s = \int_0^1 \sqrt{1 + 9x}\, dx$

V. $u = 1 + 9x$, $du = 9dx$

$x = 0 \to u = 1$

$x = 1 \to u = 10$

$$s = \frac{1}{9}\int_1^{10} \sqrt{u}\, du = \frac{1}{9}\,\frac{2}{3}\, u^{3/2}\Big|_1^{10}$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right)$$

8.4 Arc Length and Surface Area Readings

In previous sections we have used integration to answer the following questions:

1. Given a region, what is its area?
2. Given a solid, what is its volume?

In this section, we address a related question: Given a curve, what is its length? This is often referred to as **arc length**.

Consider the graph of $y = \sin x$ on $[0, \pi]$ given in Figure 8.4.1(a). How long is this curve? That is, if we were to use a piece of string to exactly match the shape of this curve, how long would the string be?

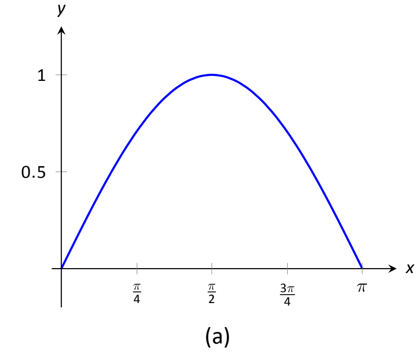


Figure 8.4.1: Graphing $y = \sin x$ on $[0, \pi]$ and approximating the curve with line segments.

As we have done in the past, we start by approximating; later, we will refine our answer using limits to get an exact solution.

The length of straight–line segments is easy to compute using the Distance Formula. We can approximate the length of the given curve by approximating the curve with straight lines and measuring their lengths.

In Figure 8.4.1(b), the curve $y = \sin x$ has been approximated with 4 line segments (the interval $[0, \pi]$ has been divided into 4 equally–lengthed subintervals). It is clear that these four line segments approximate $y = \sin x$ very well on the first and last subinterval, though not so well in the middle. Regardless, the sum of the lengths of the line segments is 3.79, so we approximate the arc length of $y = \sin x$ on $[0, \pi]$ to be 3.79.

In general, we can approximate the arc length of $y = f(x)$ on $[a, b]$ in the following manner. Let $a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b$ be a partition of $[a, b]$ into n subintervals. Let Δx_i represent the length of the i^{th} subinterval $[x_i, x_{i+1}]$.

Figure 8.4.2 zooms in on the i^{th} subinterval where $y = f(x)$ is approximated by a straight line segment. The dashed lines show that we can view this line segment as the hypotenuse of a right triangle whose sides have length Δx_i and Δy_i. Using the Pythagorean Theorem, the length of this line segment is $\sqrt{\Delta x_i^2 + \Delta y_i^2}$. Summing over all subintervals gives an arc length approximation

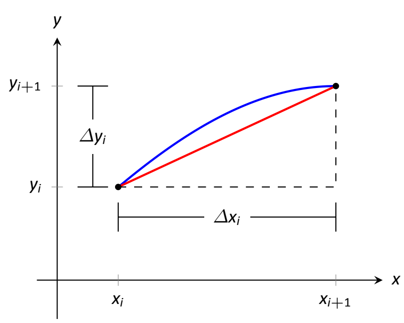


Figure 8.4.2: Zooming in on the i^{th} subin-terval $[x_i, x_{i+1}]$ of a partition of $[a, b]$.

$$L \doteq \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

As shown here, this is *not* a Riemann Sum. While we could conclude that taking a limit as the subinterval length goes to zero gives the exact arc length, we would not be able to compute the answer with a definite integral. We need first to do a little algebra.

8.4 C Arc Length and Surface Area

In the above expression factor out a Δx_i^2 term:

$$\sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 \left(1 + \frac{\Delta y_i^2}{\Delta x_i^2}\right)}.$$

Now pull the Δx_i^2 term out of the square root:

$$= \sum_{i=1}^{n} \sqrt{1 + \frac{\Delta y_i^2}{\Delta x_i^2}}\ \Delta x_i.$$

This is nearly a Riemann Sum. Consider the $\Delta y_i^2/\Delta x_i^2$ term. The expression $\Delta y_i/\Delta x_i$ measures the "change in y/change in x," that is, the "rise over run" of f on the i^{th} subinterval. The Mean Value Theorem of Differentiation (Theorem 3.2.1) states that there is a c_i in the i^{th} subinterval where $f'(c_i) = \Delta y_i/\Delta x_i$. Thus we can rewrite our above expression as:

$$= \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\ \Delta x_i.$$

This *is* a Riemann Sum. As long as f' is continuous, we can invoke Theorem 5.3.2 and conclude

$$= \int_a^b \sqrt{1 + f'(x)^2}\ dx.$$

Theorem 8.4.1 Arc Length

Let f be differentiable on $[a, b]$, where f' is also continuous on $[a, b]$. Then the arc length of f from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ dx.$$

Note: This is our first use of differentiability on a closed interval since Section 5.2.

The theorem also requires that f' be con-tinuous on $[a, b]$; while examples are ar-cane, it is possible for f to be differen-tiable yet f' is not continuous.

As the integrand contains a square root, it is often difficult to use the formula in Theorem 8.4.1 to find the length exactly. When exact answers are difficult to come by, we resort to using numerical methods of approximating definite integrals. The following examples will demonstrate this.

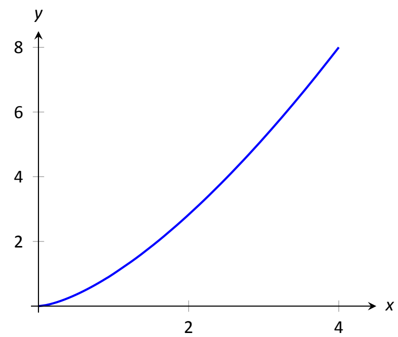


Figure 8.4.3: A graph of $f(x) = x^{3/2}$ from Example 8.4.1.

Example 8.4.1 Finding arc length

Find the arc length of $f(x) = x^{3/2}$ from $x = 0$ to $x = 4$.

Solution We find $f'(x) = \frac{3}{2}x^{1/2}$; note that on $[0, 4]$, f is differentiable and f' is also continuous. Using the formula, we find the arc length L as

$$\begin{aligned} L &= \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx \\ &= \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx \\ &= \int_0^4 \left(1 + \frac{9}{4}x\right)^{1/2}\, dx \\ &= \frac{2}{3} \cdot \frac{4}{9} \cdot \left(1 + \frac{9}{4}x\right)^{3/2}\Bigg|_0^4 \\ &= \frac{8}{27}\left(10^{3/2} - 1\right) \doteq 9.07\text{units}. \end{aligned}$$

A graph of f is given in Figure 8.4.3.

Example 8.4.2 Finding arc length

Find the arc length of $f(x) = \frac{1}{8}x^2 - \ln x$ from $x = 1$ to $x = 2$.

Solution This function was chosen specifically because the resulting integral can be evaluated exactly. We begin by finding $f'(x) = x/4 - 1/x$. The arc length is

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2}\, dx \\ &= \int_1^2 \sqrt{1 + \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2}}\, dx \\ &= \int_1^2 \sqrt{\frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2}}\, dx \\ &= \int_1^2 \sqrt{\left(\frac{x}{4} + \frac{1}{x}\right)^2}\, dx \end{aligned}$$

$$= \int_1^2 \left(\frac{x}{4} + \frac{1}{x}\right)\,dx$$
$$= \left(\frac{x^2}{8} + \ln x\right)\Bigg|_1^2$$
$$= \frac{3}{8} + \ln 2 \doteq 1.07 \text{ units.}$$

A graph of f is given in Figure 8.4.4; the portion of the curve measured in this problem is in bold.

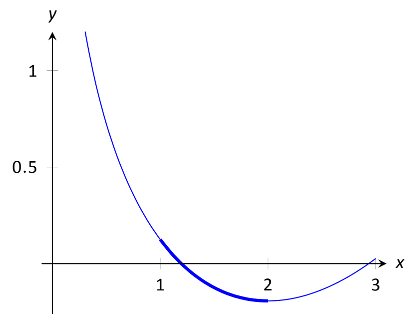


Figure 8.4.4: A graph of $f(x) = \frac{1}{8}x^2 - \ln x$ from Example 8.4.2.

The previous examples found the arc length exactly through careful choice of the functions. In general, exact answers are much more difficult to come by and numerical approximations are necessary.

Example 8.4.3 Approximating arc length numerically

Find the length of the sine curve from $x = 0$ to $x = \pi$.

Solution This is somewhat of a mathematical curiosity; in Example 5.4.3 we found the area under one "hump" of the sine curve is 2 square units; now we are measuring its arc length.

The setup is straightforward: $f(x) = \sin x$ and $f'(x) = \cos x$. Thus

$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx.$$

This integral *cannot* be evaluated in terms of elementary functions so we will approximate it with Simpson's Method with $n = 4$. Figure 7.4.5 gives $\sqrt{1+\cos^2 x}$ evaluated at 5 evenly spaced points in $[0, \pi]$. Simpson's Rule then states that

$$\int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx \doteq \frac{\pi - 0}{4 \cdot 3}\left(\sqrt{2} + 4\sqrt{3/2} + 2(1) + 4\sqrt{3/2} + \sqrt{2}\right)$$
$$= 3.82918.$$

Using a computer with $n = 100$ the approximation is $L \doteq 3.8202$; our approximation with $n = 4$ is quite good.

x	$\sqrt{1+\cos^2 x}$
0	$\sqrt{2}$
$\pi/4$	$\sqrt{3/2}$
$\pi/2$	1
$3\pi/4$	$\sqrt{3/2}$
π	$\sqrt{2}$

Figure 8.4.5: A table of values of

$y = \sqrt{1+\cos^2 x}$

to evaluate a definite integral in

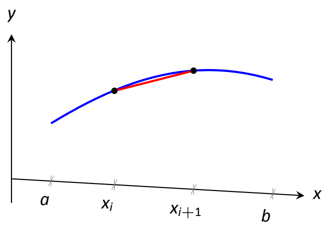
Example 8.4.3.

Surface Area of Solids of Revolution

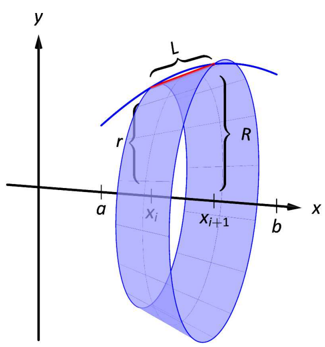
We have already seen how a curve $y = f(x)$ on $[a, b]$ can be revolved around an axis to form a solid. Instead of computing its volume, we now consider its surface area.

We begin as we have in the previous sections: we partition the interval $[a, b]$ with n subintervals, where the i^{th} subinterval is $[x_i, x_{i+1}]$. On each subinterval, we can approximate the curve $y = f(x)$ with a straight line that connects $f(x_i)$ and $f(x_{i+1})$ as shown in Figure 8.4.6(a). Revolving this line segment about the x-axis creates part of a cone (called a *frustum* of a cone) as shown in Figure 8.4.6(b). The surface area of a frustum of a cone is

$$2\pi \cdot \text{ length } \cdot \text{average of the two radii } R \text{ and } r.$$



(a)



(b)

Figure 8.4.6: Establishing the formula for surface area.

The length is given by L; we use the material just covered by arc length to state that

$$L \doteq \sqrt{1 + f'(c_i)^2}\Delta x_i$$

for some c_i in the i^{th} subinterval. The radii are just the function evaluated at the endpoints of the interval. That is,

$$R = f(x_{i+1}) \quad \text{and} \quad r = f(x_i).$$

Thus the surface area of this sample frustum of the cone is approximately

$$2\pi\frac{f(x_i) + f(x_{i+1})}{2}\sqrt{1 + f'(c_i)^2}\Delta x_i.$$

Since f is a continuous function, the Intermediate Value Theorem states there is some d_i in $[x_i, x_{i+1}]$ such that $f(d_i) = \dfrac{f(x_i) + f(x_{i+1})}{2}$; we can use this to rewrite the above equation as

$$2\pi f(d_i)\sqrt{1 + f'(c_i)^2}\Delta x_i.$$

Summing over all the subintervals we get the total surface area to be approximately

$$\text{Surface Area} \doteq \sum_{i=1}^{n} 2\pi f(d_i)\sqrt{1 + f'(c_i)^2}\Delta x_i,$$

which is a Riemann Sum. Taking the limit as the subinterval lengths go to zero gives us the exact surface area, given in the following theorem.

Theorem 8.4.2 Surface Area of a Solid of Revolution

Let f be differentiable on $[a, b]$, where f' is also continuous on $[a, b]$.

1. The surface area of the solid formed by revolving the graph of $y = f(x)$, where $f(x) \geq 0$, about the x-axis is

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx.$$

2. The surface area of the solid formed by revolving the graph of $y = f(x)$ about the y-axis, where $a, b \geq 0$, is

$$\text{Surface Area} = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\,dx.$$

(When revolving $y = f(x)$ about the y-axis, the radii of the resulting frustum are x_i and x_{i+1}; their average value is simply the midpoint of the interval. In the limit, this midpoint is just x. This gives the second part of Theorem 8.4.2.)

Example 8.4.4 Finding surface area of a solid of revolution
Find the surface area of the solid formed by revolving $y = \sin x$ on $[0, \pi]$ around the x-axis, as shown in Figure 8.4.7.

SOLUTION The setup is relatively straightforward. Using Theorem 7.4.2, we have the surface area SA is:

$$\begin{aligned} SA &= 2\pi \int_0^\pi \sin x\sqrt{1 + \cos^2 x}\,dx \\ &= -2\pi\frac{1}{2}\left(\sinh^{-1}(\cos x) + \cos x\sqrt{1 + \cos^2 x}\right)\Big|_0^\pi \\ &= 2\pi\left(\sqrt{2} + \sinh^{-1} 1\right) \doteq 14.42 \text{ units}^2. \end{aligned}$$

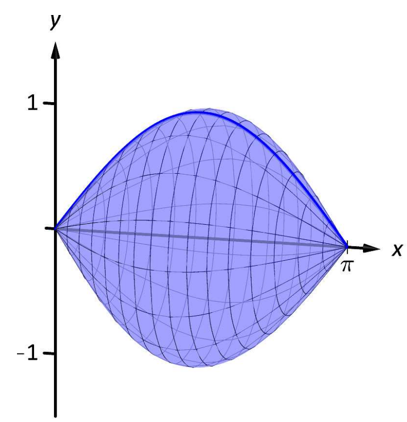


Figure 8.4.7: Revolving $y = \sin x$ on $[0, \pi]$ about the x-axis.

The integration step above is nontrivial, utilizing an integration method called Trigonometric Substitution.

It is interesting to see that the surface area of a solid, whose shape is defined by a trigonometric function, involves both a square root and an inverse hyperbolic trigonometric function.

Example 8.4.5 Finding surface area of a solid of revolution
Find the surface area of the solid formed by revolving the curve $y = x^2$ on $[0, 1]$ about the x-axis and the y-axis.

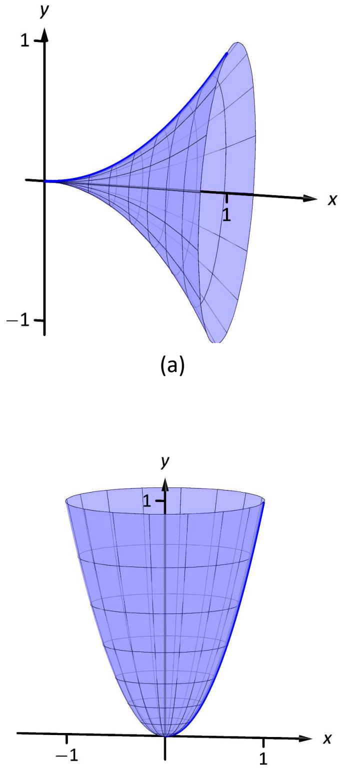

(a)
(b)

Figure 8.4.8: The solids used in Example 8.4.5.

Solution About the x-axis: the integral is straightforward to setup:

$$SA = 2\pi \int_0^1 x^2\sqrt{1+(2x)^2}\,dx.$$

Like the integral in Example 8.4.4, this requires Trigonometric Substitution.

$$\begin{aligned} &= \frac{\pi}{32}\left(2(8x^3+x)\sqrt{1+4x^2} - \sinh^{-1}(2x)\right)\Big|_0^1 \\ &= \frac{\pi}{32}\left(18\sqrt{5} - \sinh^{-1}2\right) \\ &\doteq 3.81 \text{ units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ around the x-axis is graphed in Figure 8.4.8 (a).

About the y-axis: since we are revolving around the y-axis, the "radius" of the solid is not $f(x)$ but rather x. Thus the integral to compute the surface area is:

$$SA = 2\pi \int_0^1 x\sqrt{1+(2x)^2}\,dx.$$

This integral can be solved using substitution. Set $u = 1+4x^2$; the new bounds are $u = 1$ to $u = 5$. We then have

$$\begin{aligned} &= \frac{\pi}{4}\int_1^5 \sqrt{u}\,du \\ &= \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big|_1^5 \\ &= \frac{\pi}{6}\left(5\sqrt{5}-1\right) \\ &\doteq 5.33 \text{ units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ about the y-axis is graphed in Figure 8.4.8 (b).

Our final example is a famous mathematical "paradox."

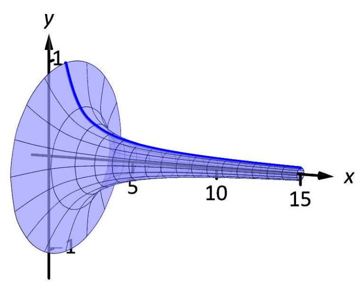


Figure 8.4.9: A graph of Gabriel's Horn.

Example 8.4.6 The surface area and volume of Gabriel's Horn

Consider the solid formed by revolving $y = 1/x$ about the x-axis on $[1, \infty)$. Find the volume and surface area of this solid. (This shape, as graphed in Figure 8.4.9, is known as "Gabriel's Horn" since it looks like a very long horn that only a supernatural person, such as an angel, could play.)

SOLUTION To compute the volume it is natural to use the Disk Method. We have:

$$\begin{aligned} V &= \pi \int_1^\infty \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi \int_1^b \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi \left(\frac{-1}{x}\right)\Bigg|_1^b \\ &= \lim_{b\to\infty} \pi \left(1-\frac{1}{b}\right) \\ &= \pi \text{ units}^3. \end{aligned}$$

Gabriel's Horn has a finite volume of π cubic units. Since we have already seen that regions with infinite length can have a finite area, this is not too difficult to accept.

We now consider its surface area. The integral is straightforward to setup:

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

Integrating this expression is not trivial. We can, however, compare it to other improper integrals. Since $1 < \sqrt{1+1/x^4}$ on $[1,\infty)$, we can state that

$$2\pi \int_1^\infty \frac{1}{x}\,dx < 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

By Key Idea 6.8.1, the improper integral on the left diverges. Since the integral on the right is larger, we conclude it also diverges, meaning Gabriel's Horn has infinite surface area.

Hence the "paradox": we can fill Gabriel's Horn with a finite amount of paint, but since it has infinite surface area, we can never paint it.

Somehow this paradox is striking when we think about it in terms of volume and area. However, we have seen a similar paradox before, as referenced above. We know that the area under the curve $y = 1/x^2$ on $[1,\infty)$ is finite, yet the shape has an infinite perimeter. Strange things can occur when we deal with the infinite.

A standard equation from physics is "Work = force $\times$ distance", when the force applied is constant. In the next section we learn how to compute work when the force applied is variable.

NOTE No real paradox here at all. When you paint the area under the curve, the paint thickness is the same for all x. Infinite area ⇒ infinite volume. When you fill the horn with paint, its thickness decreases in two dimensions:

"small x small = very small"

Exercises

Again, it is an excellent exercise to translate each of the examples of Apex above into the 5 step Method.

Terms and Concepts

1. T/F: The integral formula for computing Arc Length was found by first approximating arc length with straight line segments.

2. T/F: The integral formula for computing Arc Length includes a square–root, meaning the integration is probably easy.

Problems

In Exercises 3 – 12, find the arc length of the function on the given interval.

3. $f(x) = x$ on $[0, 1]$.

4. $f(x) = \sqrt{8}x$ on $[-1, 1]$.

5. $f(x) = \dfrac{1}{3}x^{3/2} - x^{1/2}$ on $[0, 1]$.

6. $f(x) = \dfrac{1}{12}x^3 + \dfrac{1}{x}$ on $[1, 4]$.

7. $f(x) = 2x^{3/2} - \dfrac{1}{6}\sqrt{x}$ on $[0, 9]$.

8. $f(x) = \cosh x$ on $[-\ln 2, \ln 2]$.

9. $f(x) = \dfrac{1}{2}\left(e^x + e^{-x}\right)$ on $[0, \ln 5]$.

10. $f(x) = \dfrac{1}{12}x^5 + \dfrac{1}{5x^3}$ on $[.1, 1]$.

11. $f(x) = \ln\left(\sin x\right)$ on $[\pi/6, \pi/2]$.

12. $f(x) = \ln\left(\cos x\right)$ on $[0, \pi/4]$.

In Exercises 13 – 20, set up the integral to compute the arc length of the function on the given interval. Do not evaluate the integral.

13. $f(x) = x^2$ on $[0, 1]$.

14. $f(x) = x^{10}$ on $[0, 1]$.

15. $f(x) = \sqrt{x}$ on $[0, 1]$.

16. $f(x) = \ln x$ on $[1, e]$.

17. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: this describes the top half of a circle with radius 1.)

18. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: this describes the top half of an ellipse with a major axis of length 6 and a minor axis of length 2.)

19. $f(x) = \dfrac{1}{x}$ on $[1, 2]$.

20. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 21 – 28, use Simpson's Rule, with $n = 4$, to approximate the arc length of the function on the given interval. Note: these are the same problems as in Exercises 13–20.

21. $f(x) = x^2$ on $[0, 1]$.

22. $f(x) = x^{10}$ on $[0, 1]$.

23. $f(x) = \sqrt{x}$ on $[0, 1]$. (Note: $f'(x)$ is not defined at $x = 0$.)

24. $f(x) = \ln x$ on $[1, e]$.

25. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: $f'(x)$ is not defined at the endpoints.)

26. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: $f'(x)$ is not defined at the endpoints.)

27. $f(x) = \dfrac{1}{x}$ on $[1, 2]$.

28. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 29 – 33, find the surface area of the described solid of revolution.

29. The solid formed by revolving $y = 2x$ on $[0, 1]$ about the x-axis.

30. The solid formed by revolving $y = x^2$ on $[0, 1]$ about the y-axis.

31. The solid formed by revolving $y = x^3$ on $[0, 1]$ about the x-axis.

32. The solid formed by revolving $y = \sqrt{x}$ on $[0, 1]$ about the x-axis.

33. The sphere formed by revolving $y = \sqrt{1 - x^2}$ on $[-1, 1]$ about the x-axis.

Solutions 8.4

1. T
2. F
3. $\sqrt{2}$
4. 6
5. $4/3$
6. 6
7. $109/2$
8. $3/2$
9. $12/5$
10. $79953333/400000 \doteq 199.883$
11. $-\ln(2-\sqrt{3}) \doteq 1.31696$
12. $\sinh^{-1} 1$
13. $\int_0^1 \sqrt{1+4x^2}\, dx$
14. $\int_0^1 \sqrt{1+100x^{18}}\, dx$
15. $\int_0^1 \sqrt{1+\frac{1}{4x}}\, dx$
16. $\int_1^e \sqrt{1+\frac{1}{x^2}}\, dx$
17. $\int_{-1}^1 \sqrt{1+\frac{x^2}{1-x^2}}\, dx$
18. $\int_{-3}^3 \sqrt{1+\frac{x^2}{81-9x^2}}\, dx$
19. $\int_1^2 \sqrt{1+\frac{1}{x^4}}\, dx$
20. $\int_{-\pi/4}^{\pi/4} \sqrt{1+\sec^2 x \tan^2 x}\, dx$
21. 1.4790
22. 1.8377
23. Simpson's Rule fails, as it requires one to divide by 0. However, recognize the answer should be the same as for $y = x^2$; why?
24. 2.1300
25. Simpson's Rule fails.
26. Simpson's Rule fails.
27. 1.4058
28. 1.7625
29. $2\pi \int_0^1 2x\sqrt{5}\, dx = 2\pi\sqrt{5}$
30. $2\pi \int_0^1 x\sqrt{1+4x^2}\, dx = \pi/6(5\sqrt{5}-1)$
31. $2\pi \int_0^1 x^3\sqrt{1+9x^4}\, dx = \pi/27(10\sqrt{10}-1)$
32. $2\pi \int_0^1 \sqrt{x}\sqrt{1+1/(4x)}\, dx = \pi/6(5\sqrt{5}-1)$
33. $2\pi \int_0^1 \sqrt{1-x^2}\sqrt{1+x/(1-x^2)}\, dx = 4\pi$

8.5 Water Tank Problems

There are many work problem in physics and engineering, often involving advanced scientific concepts.

One easy to understand application is calculating the work required to fill a water tank. It is more complicated in some ways than the previous applications of integrations in that two asymptotic equality approximations are often required. Recall:

Theorem $a \approx A,\ b \approx B \iff a \cdot A \approx b \cdot B$

The most basic work problem in grade 10 physics is moving a through a straight line distance d by a constant force F is

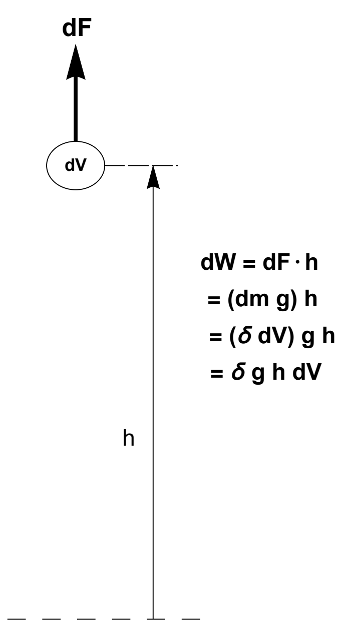
$$W = F \cdot d$$

A water tank problem is more complicated. An infinitesimal element of volume dV meter^3 is lifted by *a* force $dF = dm \cdot g$ where $g = 9.08$ meter/sec^2 is the acceleration of gravity acting through *a* vertical distance *d* meters.

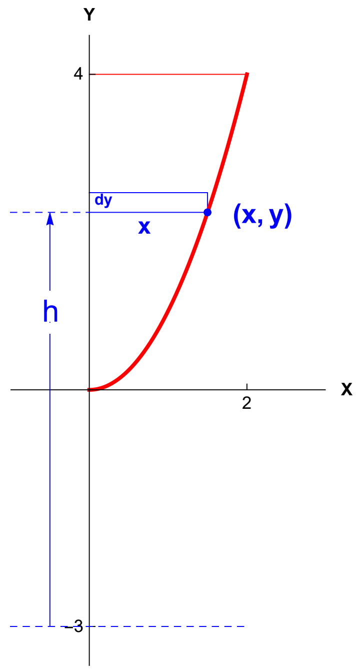
The infinitesimal work done in lifting the element is then

$$dW = df \cdot h = (dm \cdot g)\, h = (\delta\, dV)\, g\, h = \delta\, g\, h\, dV.$$

For water, $\delta = 1000\ \frac{Kg}{m^3}$.



Example A water tank is made by rotating the curve $y = x^2$ meters, $0 \le x \le 2$ meters about the Y-axis. How much work is required to fill the tank from a source 3 meters below the bottom of the tank?



Note that in these problems
there are two approximations:

radius of element $\approx$ x
h $\approx$ y + 3
and we must use

Theorem $A \approx B,\ C \approx D \Longrightarrow AC \approx BD$

$$\begin{aligned} dV &\approx \pi x^2\, dy = \pi y\, dy \\ h &\approx y - (-3) = y + 3 \\ dW &\approx \delta g h\, dV \\ &= \delta g (y+3)(\pi y\, dy) \\ &= \delta g \pi (y^2 + 3y)\, dy \end{aligned}$$

$$\Rightarrow W = \delta g \pi \int_0^4 (y^2 + 2y)\, dy = \delta g \pi \left[\frac{y^3}{3} + \tfrac{3}{2} y^2 \right|_0^4 \text{ Joules}$$

Use the *Five Step Procedure* in each problem

Exercises Use the grade 10 formula **$dW = \delta\, g\, h\, dV$** in each problem.

#1. A water tank is 5 meters high and has a square cross section 2 meters on a side.

a. How much work is required to fill the tank from a well 10 meters below the bottom of the tank?

b. How much work is required to empty the full tank to a height 10 meters above the top of the tank?

#2. A water tank is made by rotating the curve $y = x$ meters, $0 \le x \le 2$ meters about the Y-axis.

a. How much work is required to fill the tank from the bottom of the tank?

b. How much work is required to empty the full tank over the top of the tank?

#3. A water tank is 5 meters high and has a circular cross section of radius 1 meter.

a. How much work is required to fill a half full tank from a well 10 meters below the bottom of the tank?

b. How much work is required to half empty a full tank to a height 10 meters above the top of the tank?

#4. A water tank is 10 meters long and has an equilateral triangle cross section of side 2 meters, point down.

a. How much work is required to fill the tank from the bottom of the tank?

b. How much work is required to empty the full tank out over the top of the tank?

#5. A water tank lying on its side is 5 meters long and has a circular cross section of radius 1 meter.

a. How much work is required to fill the tank from a well 4 meters below the bottom of the tank?

b. How much work is required to empty a full tank to a height 4 meters above the top of the tank?

#6. A water tank is a sphere of radius 2 meters.

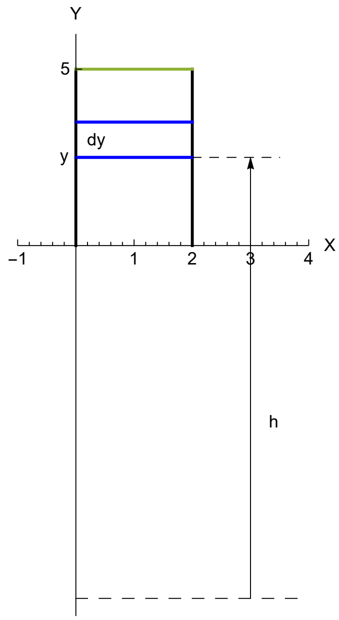
a. How much work is required to fill the tank from a well 5 meters below the bottom of the tank?

b. How much work is required to empty a full tank to a height 5 meters above the top of the tank?

#7. How much work is required to fill the tank of Exercise 23 of section 2 in this chapter with water pumped in at the bottom of the tank?

Solutions

#1a.



$h = y + 10$

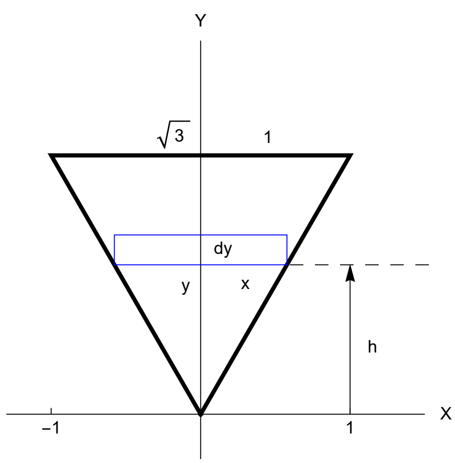
$dV = 2^2\, dy = 4\, dy$

$dW = \delta\, g\, h\, dV = 4\delta\, g\,(y + 10)dy$

$W = 4\delta g \int_0^5 (y + 10)\, dy$

$= 250\, \delta\, g$

#4a.



$h = y$

By similar triangles

$\frac{x}{1} = \frac{y}{\sqrt{3}}$

$y = \sqrt{3}\, x$

$dV = 2x \cdot 10\, dy$

$= 20x\, dy$

$= \frac{20}{\sqrt{3}} y\, dy$

$dW = \delta\, g\, h\, dV = \frac{20}{\sqrt{3}} \delta\, g\, y^2\, dy$

$W = \frac{20}{\sqrt{3}} \delta\, g \int_0^{\sqrt{3}} y^2\, dy$

$= 20\, \delta\, g$

8.6 Application to Economics. Present and Future Value

A dollar at a future time is not worth as much as a dollar now, because a dollar now can be invested at the going interest rate and so will be worth more than one dollar at that future time.

We will assume in the calculations of this section that whether borrowing or investing, money is worth a constant going interest rate r over the period of time considered. The interest rate can include a component that compensates for inflation.

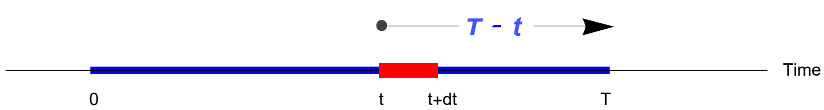
Discrete Investment or Income, Lump Sum Recall the exponential lump sum growth formula

$$F = P e^{rt}.$$

$F = P e^{rt}$	**future value of *a* present sum**
Solving for P:	
$P = F e^{-rt}$	**present value of *a* future sum**

Continuous Investments or Income Suppose an investment is made or an income is received continuously according or an ***investment /income stream*** $I = I(t) \frac{\$}{\text{year}}$. In the following derivation, we assume that I(t) is a continuous function and therefore approximately constant on the interval of length dt; then the amount of money received/invested during that interval is approximately

'rate × time' ≈ I(t) dt.



Total Money The amount invested/received on the interval dt:

$dM \approx I(t)\, dt.$

So the total money invested/received is

$$M = \int_0^T I(t)\, dt.$$

Present Value The present value of the money on the interval dt at time 0, by the discrete formula

$dP \approx dM\, e^{-rt} = I(t)\, e^{-rt}\, dt$

So the total present value is

$$P = \int_0^T I(t)\, e^{-rt}\, dt$$

Future Value The future value of the money on the interval dt at time T, by the discrete formula (1):

$dF \approx dM\, e^{r(T-t)} = I(t)\, e^{r(T-t)}\, dt$

So the total future value is

$$F = \int_0^T I(t)\, e^{r(T-t)}\, dt$$

Use the *Five Step Procedure* in each problem

Exercises

For each of #1 to 7 find:

a. The total money (received or invested)

b. The present value of this money.

c. The future value of this money.

#0. You invest $10000 lump sum now at an interest rate of 10% for 40 years.

#1. You invest $1000 per year at an interest rate of 10% for 40 years.

#2. You invest $100 t per year at an interest rate of 10% for 40 years.

#3. You invest $10 t^2 per year at an interest rate of 10% for 40 years

For each of $4 to 7 also determine how much should you pay for the annuity. Money is worth 10% interest.

#4. You purchase an annuity which pays 10000 $/year for 20 years.

#5. You purchase an annuity which pays 10000 $/year forever.

#6. You purchase an annuity which pays 1000 t $/year for 20 years.

#7. You purchase an annuity which pays 1000 t $/year forever.

Answer each of the following two questions. You wish to give you child $100,000 in 20 years for its education. You invest at 10%/year return.

#8. What lump sum should you invest now?

#9. At what constant yearly rate should you invest over the next 20 years?

10. Explain why interest implies inflation and why some of the world's major religions have at times forbidden interest.

Solutions/ Hints

#0. M = total money invested = $ 10,000

P = present value = $ 10,000

F = future value = $10000\, e^{0.1\,(40)} \doteq$ $ 545,982

#1. M = $\int_0^{40} 1000\, \mathrm{dt}$

P = $\int_0^{40} 1000\, e^{-0.1\,t}\, \mathrm{dt}$

F = $\int_0^{40} 1000\, e^{0.1\,(40-t)}\, \mathrm{dt}$

#2. I(t) = 100t

M = $\int_0^{40} 100\, t\, \mathrm{dt}$

P = $\int_0^{40} 100\, t e^{-0.1\,t}\, \mathrm{dt}$

F = $\int_0^{40} 100\, t e^{0.1\,(40-t)}\, \mathrm{dt}$

#4. P = $\int_0^{20} 10\,000\, e^{-0.1\,t}\, \mathrm{dt}$

#5. The present value, what you should pay.

P = $\int_0^{+\infty} 10\,000\, e^{-0.1\,t}\, \mathrm{dt}$

= $100,000

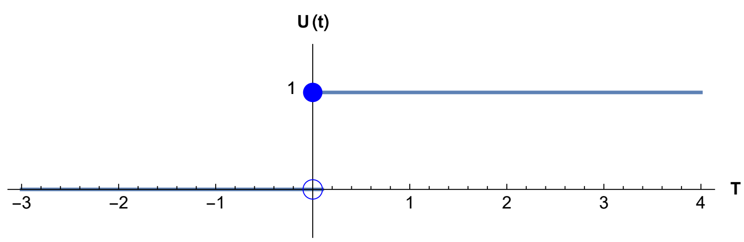
Chapter 9 Generalized Functions

Generalized functions and their special generalized calculus give correct answers in some important areas of application where ordinary calculus fails. **One area is writing piecewise defined functions in a form where the Fundamental Theorem of Calculus applies.** **Another is a way of representing impulse spikes or the density of a point particle in function form.**

9.1 Piecewise Defined Functions

In applications, functions are often defined by piecing together simpler continuous functions.

Example The ***Unit Step Function***, U(t). This piecewise defined function is very important in applications. For example, it can represent 'turning on' an electric potential of 1 Volt exactly at time t = 0.

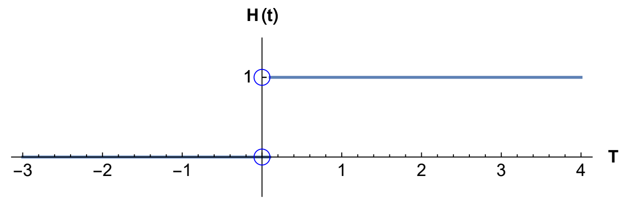


Sectionally Continuous Functions For both reasons of application and mathematics we modify acceptable types of piecewise defined functions.

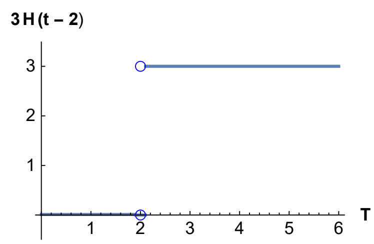
Definition A ***sectionally continuous function*** **is a function which is**

1. **continuous on the real line except at finitely many points in each subinterval.**
2. **at each point of discontinuity x_i, $f(x_i^-)$ and $f(x_i^+)$ are finite real numbers.**
   $f(x_i^+)$ is the limit from the right as x approaches x_i
   $f(x_i^-)$ is the limit from the left as x approaches x_i
3. **$f(x)$ is undefined at each point of discontinuity.**
4. **$f(-\infty) = 0$ (this is not unduly restrictive because in many application, a quantity is 0 initially or 'turned on' at some finite time after the creation).**

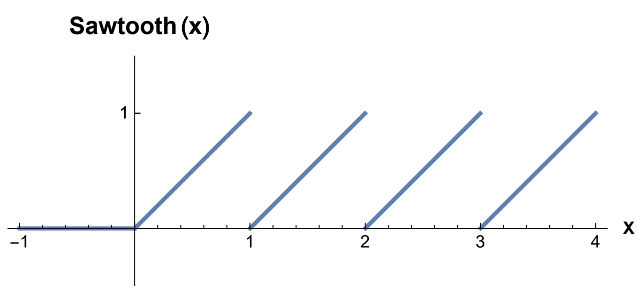
The Unit Step Function is not a sectionally continuous function according to our definition because it is defined at the jump point $t = 0$. A modification of that function, the ***Heaviside Generalized Function***, is a sectionally continuous function. In applications, it is physically meaningful not to define the function at $t = 0$ because one does not know and it does not matter whether a quantity is turned on exactly at or just before or just after $t = 0$!



Example The Heaviside function multiplied by 3 and translated 2 to the right is $3\,H(t-2)$.
It can represent 'turning on' an electric potential of 3 Volts at $t = 2$.

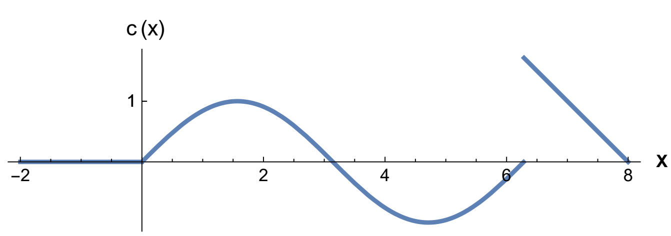


Example A sawtooth function. The open circle convention at discontinuities is normally assumed when working with generalized functions and will not usually be shown explicitly.

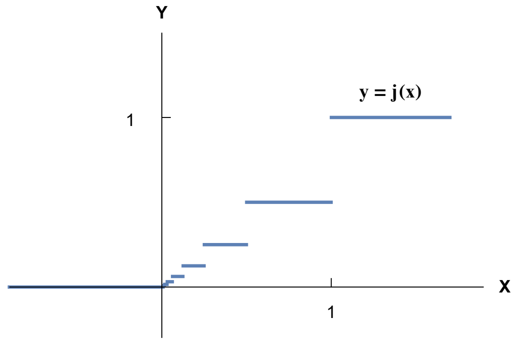


Example A more complicated sectionally continuous function is

$$c(x) = \begin{cases} 0, & x < 0 \\ \sin x, & 0 \leqslant x < 2\pi. \\ x - 8, & x > 2\pi \end{cases}$$



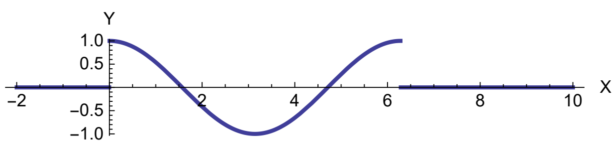
Example This jump function $j(x)$ is not a sectionally continuous function because it has infinitely many discontinuities on the finite interval $0 < x < 1$. (At endpoints of continuous subintervals, $j(x)$ is not defined.) Nevertheless, because the discontinuities occur on an appropriate sequence of points, the theory we will develop in the next sections will also apply to this example.



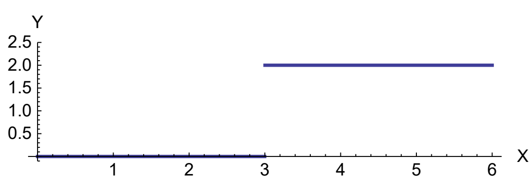
Exercises

1. Write the equation for each sectionally continuous function using piecewise notation.

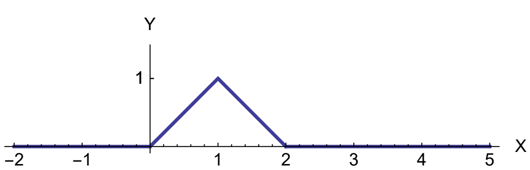
a.



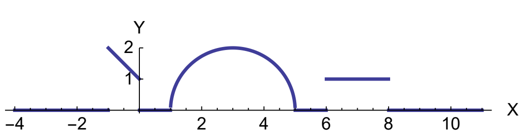
b.



c.



d.



2. Graph each. Which are sectionally continuous functions? Graph each. Modify the ones which are not sectionally continuous, when possible, so they are sectionally continuous.

a. $f(x) = \begin{cases} 0, & x < -1 \\ 2, & x \geqslant -1 \end{cases}$

b. $g(x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0 \end{cases}$

c. $h(x) = \begin{cases} 0, & x < 0 \\ \sin(\pi x), & 0 \leqslant x \leqslant 2 \\ 0, & x > 2 \end{cases}$

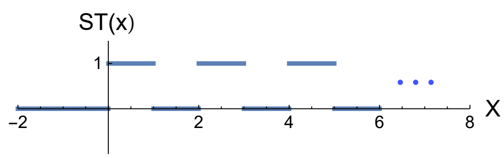
d. $k(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ -1, & 2 < x < 4 \\ 0, & x > 4 \end{cases}$

e. $l(x) = e^{x}$

e. $m(x) = e^{-x}$.

3. Invent four sectionally continuous functions. Graph each.

4. The Square Tooth Function (repeats indefinitely to the right).



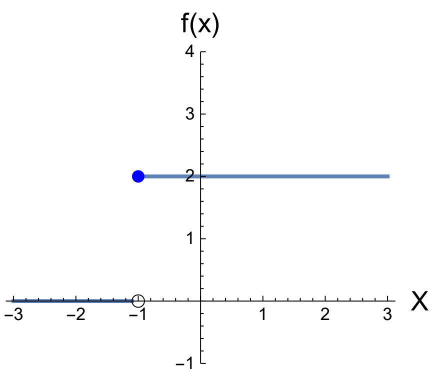
Graph each of the following. State if not a sectional continuous function.

a. $y = 3ST(x)$
b. $y = -2ST(x)$
c. $y = ST(-x)$
d. $y = ST(2x)$
e. $y = \sin x \; ST\left(\frac{x}{\pi}\right)$
f. $y = 2\,ST(x-2)$

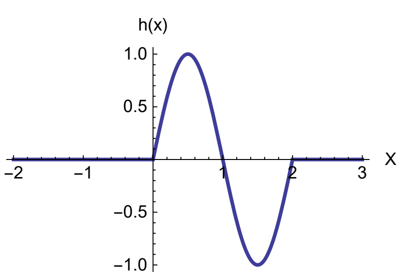
Solutions

1d. $$y = \begin{cases} 0 & x < -1 \\ 1 - x & -1 < x < 0 \\ 0 & 0 < x < 1 \\ \sqrt{4 - (x-3)^2} & 1 < x < 5 \\ 0 & 5 > x < 6 \\ 1 & 6 < x < 8 \\ 0 & x > 8 \end{cases}$$

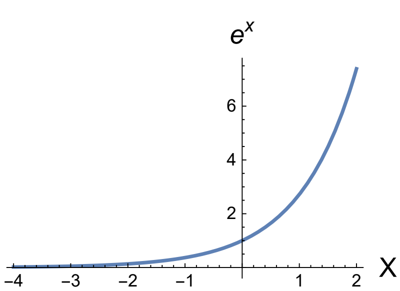
2a. not sectionally continuous.



2c. sectionally continuous



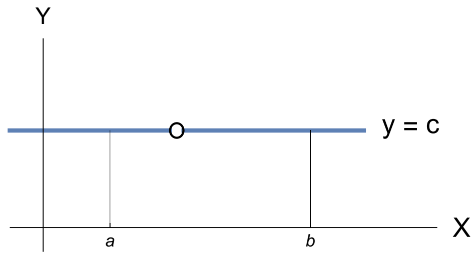
2e. sectionally continuous



9.2 Generalized Function Calculus Graphically

Generalized functions and their generalized integrals and derivatives give correct answers in many areas of applications where ordinary derivatives or integrals fail.

Generalized Integrals First we note that, by the definition of definite integral, $\int_a^b f(x)\,dx$, does not exist if f(x) is undefined somewhere on the interval a ≤ x ≤ b.



This is because in the definition of integral, $\sum_{i=1}^{N} f\,(x_i^*)\,dx \approx> \int_a^b f(x)\,dx$, if for some choice of dx, $f(x_i^*)$ is undefined, then the entire sum is undefined (even an infinitesimal defect poisons the whole sum). However, since the area under a point, no matter what's its value, is 0, we generalize the definition of definite integral to ignore isolated points where f(x) is undefined. Such an integral is called a ***generalized integral*** of f. So in this graphical example we agree that $\int_a^b f(x)\,dx = (b-a)\cdot c$. Before we continue, recall the following for reference.

Fact:

If f is continuous (except at isolated points) ⇒ $\int_a^x f\,(t)\,dt$ is smooth (differentiable). Proof: DIY

Integrating Derivatives We would like, because of the Fundamental Theorem, the following to be true:

$$\int_a^x f'(t)\,dt = f(x) - f(a),$$

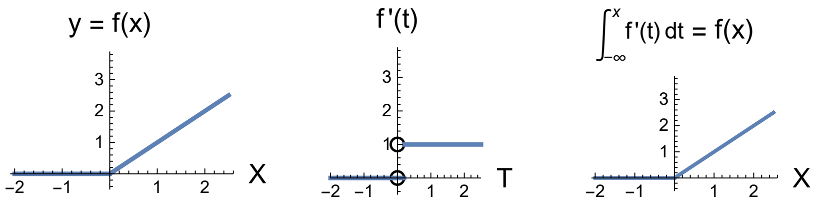
"the integral of the derivative of a function is the function minus its initial value". For sectionally functions, f(-∞) = 0. Then

$$\boxed{\int_{-\infty}^{x} f'(t)\,dt = f(x)}$$

An even better looking Fundamental Theorem!

This holds if f(x) is continuous using generalized integration. However, we will see that if f is not continuous, we will also have to generalize the idea of derivative for $\int_{-\infty}^{X} f'(t)\,dt = f(x)$ to hold.

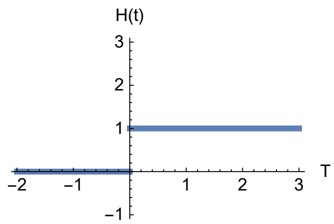
Example f(x) is continuous. ***Everything is fine***. Verify mentally, using the slope and area interpretations.



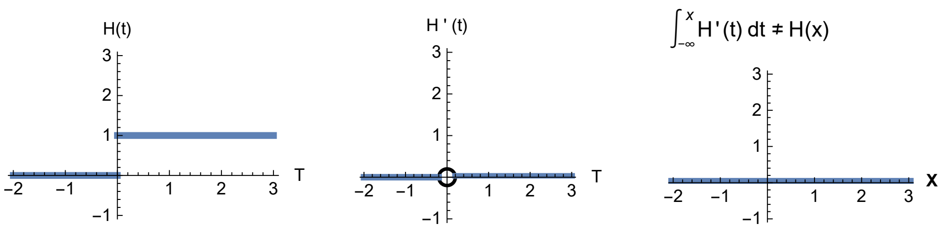
The Heaviside Function Let us recall the basic sectionally continuous function, H(x).

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}.$$

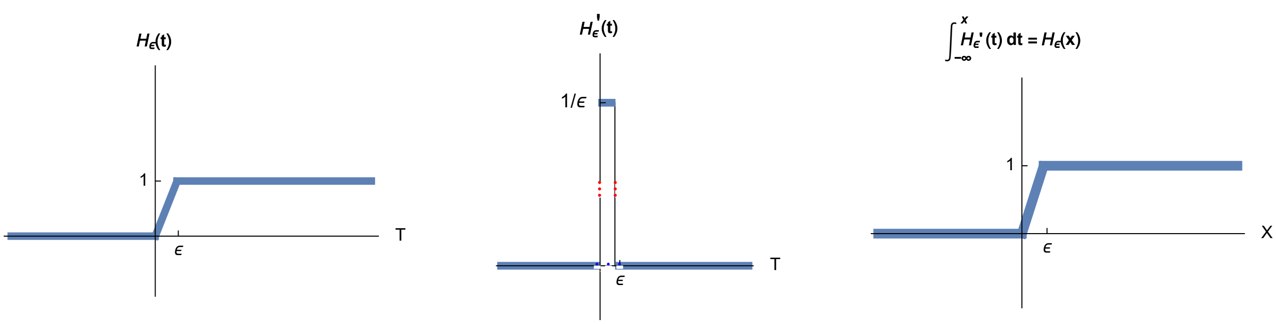
It is undefined at $t = 0$. It is useful as a multiplier in applications for 'turning on' a quantity at time $t = 0$.



Example Let us try differentiating and then integrating H(x). ***Everything is not fine***.

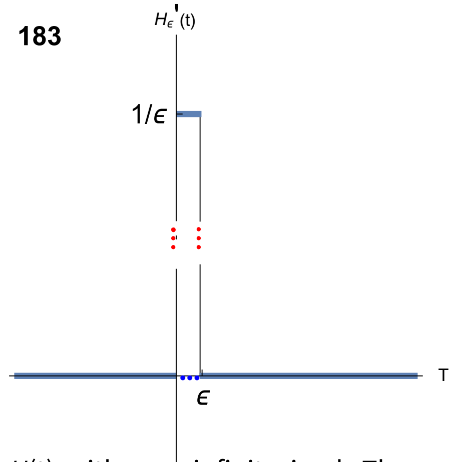


The problem in this example is taking the ordinary derivative of H(t) at a discontinuity. Somehow we lose information at $x = 0$. We will have to search for a correct 'generalized derivative'. (The generalized integral was used in integrating H '(t)). Let us search for a more useful derivative by considering a modified Heaviside function and its derivative.



In more detail:

$$H_\epsilon'(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\epsilon} & 0 < t < \epsilon \\ 0 & t > \epsilon \end{cases}$$



The above graph shows the approximation H_ϵ(t) to H(t) with ϵ an infinitesimal. The area under H_ϵ'(t) is 1, just what we need. We call the resulting idea, the ***(Dirac) delta function***, named after its inventor. It is written δ(x).

The delta function is not a function in the usual sense because at the 'interesting' place $x = 0$, it is undefined. It is called a ***distribution*** and is a ***generalized function*** .

Definition The ***Dirac Delta Function*** δ(x) is defined analytically by:

1. $\delta(x) = 0, \ x \neq 0$
2. $\int_{-\infty}^{x} \delta(t)\, dt = H(x)$. Conversely the ***generalized derivative*** of H(x) is δ(x). Equivalently $\boxed{\frac{d}{dx} H(x) = \delta(x)}$.

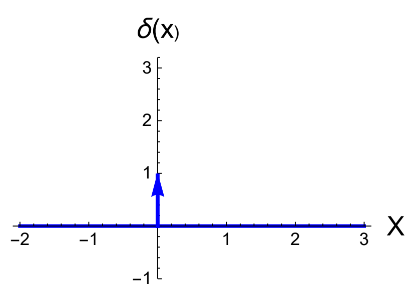
Graphically δ(x) is shown by an upward unit arrow with its tail at (0, 0).

The above H_ϵ'(t) is a perfectly good definition for δ(x) because for every infinitesimal $\epsilon > 0$ the 'spike' fits between 0 and ϵ. Also the area of the spike, 1, is independent of ϵ. However, most mathematicians prefer a definition free of infinitesimals. We will use the following definition although some proofs are easier if we use H_ϵ'(t),

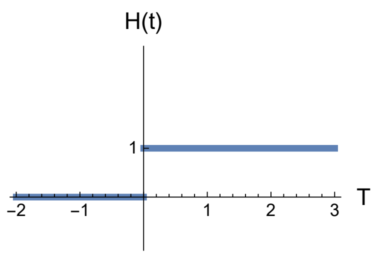
The Dirac Delta Function δ(x), a traditional definition

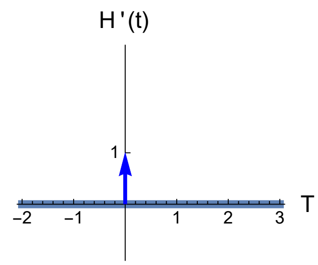
1. $\delta(x) = 0, \ x \neq 0$
2. $\int_{-\infty}^{x} \delta(t)\, dt = H(x)$ or conversely $\frac{d}{dx} H(x) = \delta(x)$

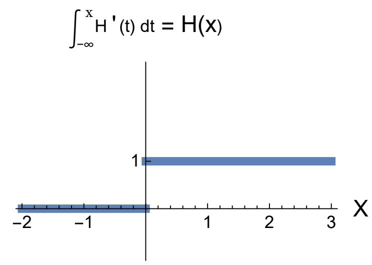
Graphically δ(x) is shown as as a unit arrow with its tail at (0, 0).



We now with generalized functions, generalized derivatives and generalized integrals have the desired derivative of H(x) which satisfies $\int_{-\infty}^{x} H'(t)\, dt = H(x)$!

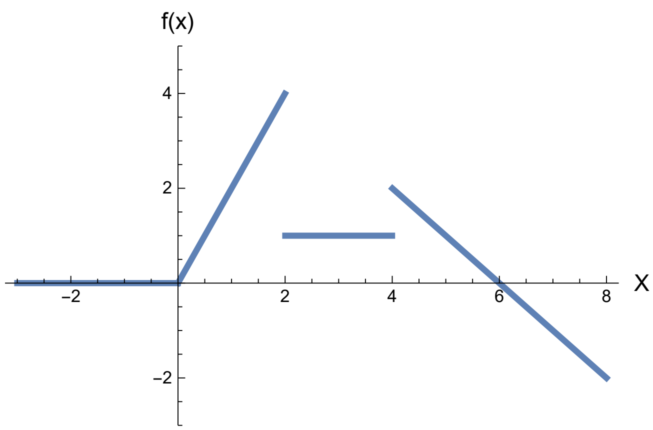




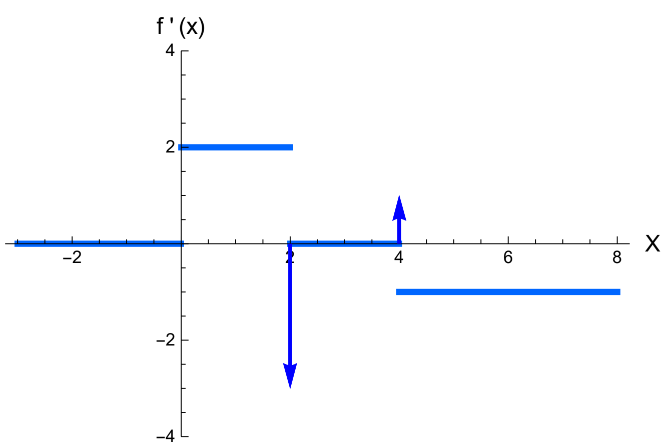


The method of finding the 'correct' generalized derivative of any generalized function is now clear. Its generalized derivative is just its ordinary derivative plus appropriated shifted delta functions multiplied by the magnitude of the function's jump at each discontinuity. We say that $b\,\delta(x\text{-}a)$ is a 'delta function of ***strength*** b at $x = a$'.

Example Let f(t) be the function graphed below.



Its generalized derivative is then:



You can verify, by visual generalized integration of f '(t) from -∞ to x, that the result is f(x).

The Rule for Graphical Generalized Differentiation:

1. **Draw the ordinary derivative of the function.**
2. **Add an arrow with length equal to the jump at each discontinuity.**

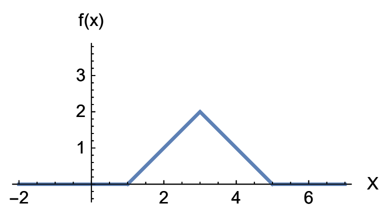
Summary Functions which are 0 for x large and negative and which are continuous and ordinary differentiable except at isolated points and are bounded are called ***generalized functions***. The ***generalized function calculus*** consists of the generalized functions and ordinary functions together with their generalized derivatives and integrals.

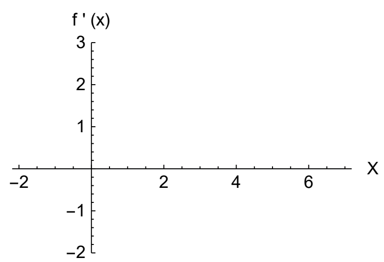
A generalized function with generalized differentiation and generalized integration satisfies the Fundamental Theorem of Calculus!

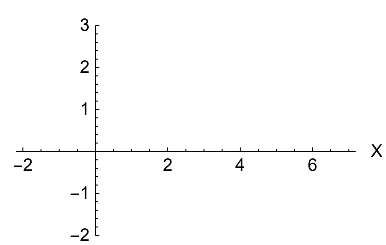
$$\int_{-\infty}^{x} f\,'(t)\,dt = f(x)$$

Exercises Do all quickly.

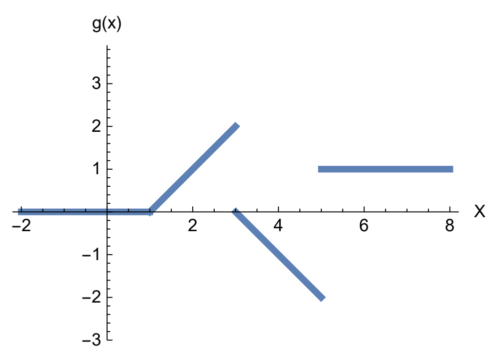
1. Graph the generalized derivative $f\,'(x)$. Verify its generalized integral is $f(x)$.

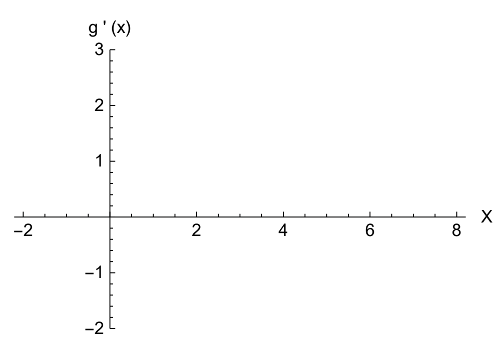




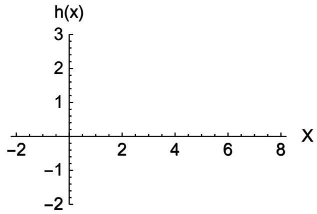


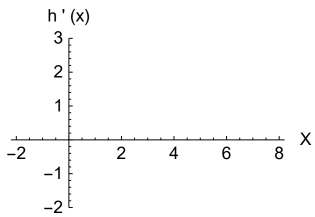
2. Graph the generalized derivative $g\,'(x)$. Verify its generalized integral is $g(x)$.

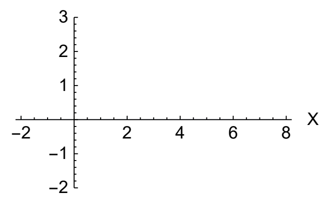




3. Invent your own h(x). Graph the generalized derivative h '(x). Verify its generalized integral is h(x).



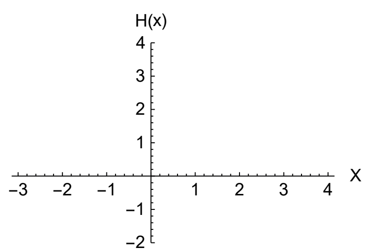


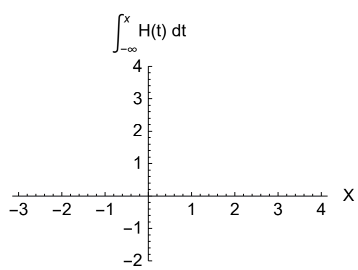


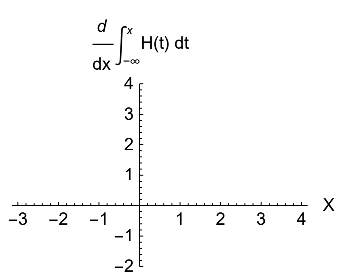
4. Generally 'integrating first and then differentiating' causes no problem: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

a. Prove this. Hint: use the Fundamental Theorem of Calculus, $\int_a^b f(t)\,dt = F(b) - F(a)$.

b. Verify this graphically for the Heaviside function H(x).



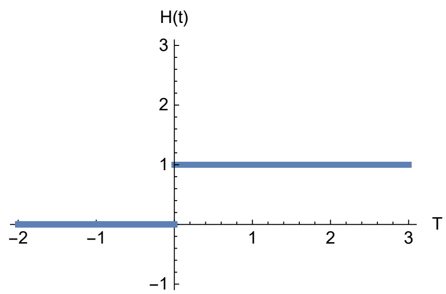




9.3 Generalized Functions Analytically

Review The Heaviside Function H(t)

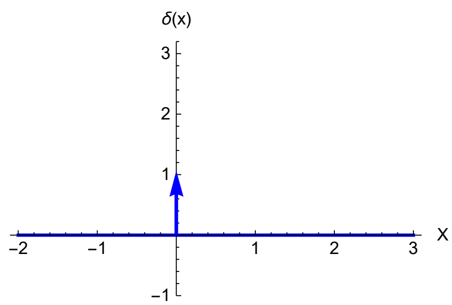
$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



Review The Dirac Delta Function δ(x):

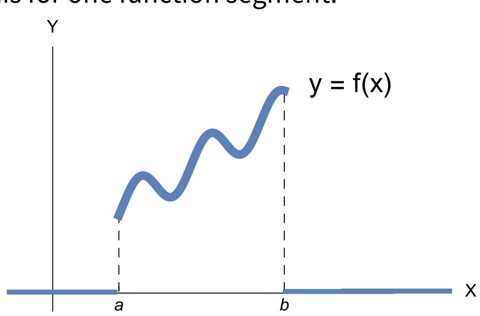
1. $\delta(x) = 0, \ x \neq 0$
2. $\int_{-\infty}^{x} \delta(t)\, dt = H(x)$ or conversely $\frac{d}{dx} H(x) = \delta(x)$

Graphically $\delta(x)$ is shown by an upward unit arrow with its tail at $x = 0$ on the X-axis.



Using the Heaviside Step function to write equations of generalized functions

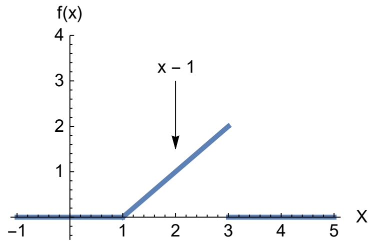
First we do this for one function segment.



y = 'turn on f(x) at x = a' and then 'turn off f(x) at x = b'
= f(x)H(x-a) - f(x)H(x-b).

For general generalized functions apply the above technique to each function segment.

Example



Its equation is

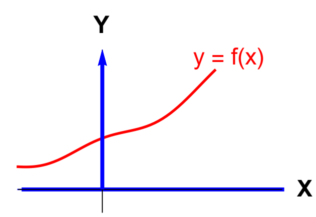
$f(x) = (x - 1)H(x\text{-}1) - (x - 1)H(x\text{-}3)$.

Generalized Calculus Properties of H and δ

1. $\delta(x) = 0,\ x \neq 0$	1. $\delta(x\text{-}a) = 0,\ x \neq a$
2. $\int_{-\infty}^{x} \delta(t)\, dt = H(x)$	2. $\int_{-\infty}^{x} \delta(t - a)\, dt = H(x\text{-}a)$
3. $H'(x) = \delta(x)$	3. $H'(x\text{-}a) = \delta(x\text{-}a)$
4. $f(x)\ \delta(x) = f(0)\ \delta(x)$	4. $f(x)\ \delta(x\text{-}a) = f(a)\ \delta(x\text{-}a)$ 'Sifting Property'

The Sifting Property is useful in simplifying expressions involving the delta function.

Proof The only mystery is #4, the Sifting Property. $f(x)\,\delta(x) = f(0)\,\delta(x)$ follows from #1, since the only value of f that 'counts' is at $x = 0$.



Example Let us look at the previous example.

$f(x) = (x - 1)H(x\text{-}1) - (x - 1)H(x\text{-}3)$.

By the Product Rule

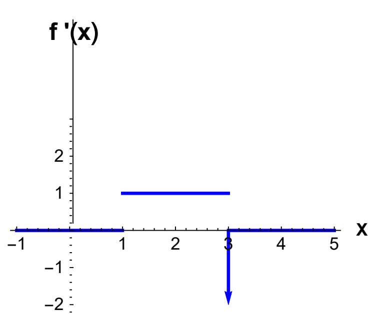
$f'(x) = 1\,H(x\text{-}1) + (x - 1)\delta(x\text{-}1) - 1\,H(x\text{-}3) - (x - 1)\delta(x\text{-}3)$

$= H(x\text{-}1) - H(x\text{-}3) - 2\delta(x\text{-}3)$ since by the Sifting Property:

$(x - 1)\delta(x\text{-}1) = (1 - 1)\delta(x\text{-}1) = 0$

$(x - 1)\delta(x\text{-}3) = (3 - 1)\delta(x\text{-}3) = 2\delta(x\text{-}3)$

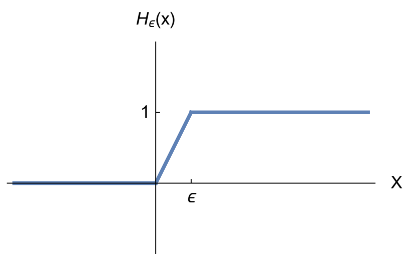
Note that this derivative does not involve a delta function at $x = 1$. This is because f is continuous there. The graph of the derivative is

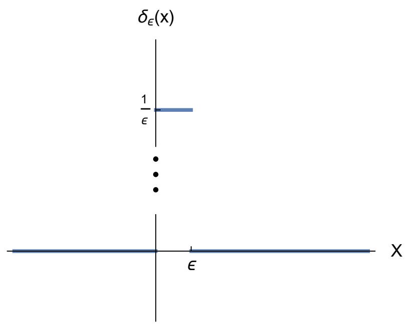


which looks just right.

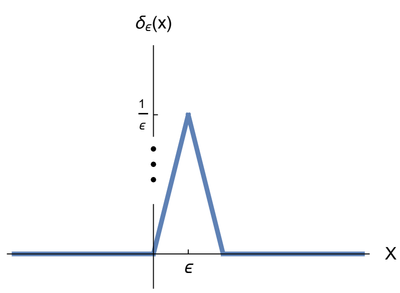
Interpretation of δ '(x) The delta function and its derivative are important in applications. Their geometric approximations are useful in understanding their properties.

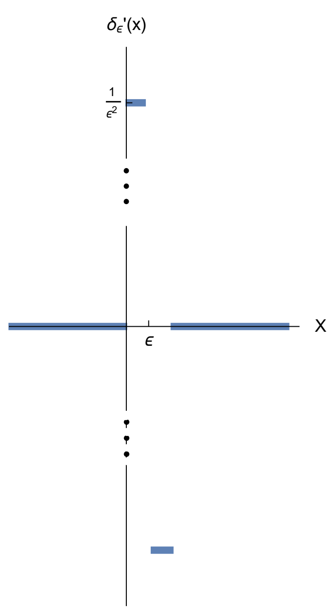
Here are the approximations of H(x) and δ(x) again and δ '(x). You can verify them by starting at the top with differentiation **or** starting at the bottom with integration.



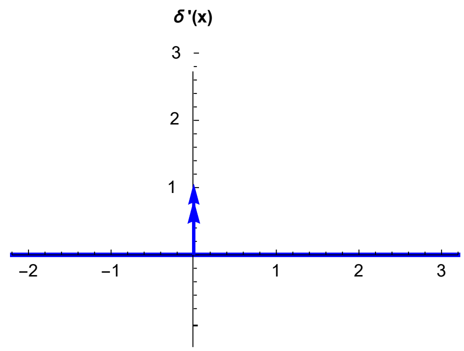


To find the derivative of $\delta_\epsilon(x)$ we will use another version of it in triangular form which also has area 1.





Graph of $\delta'(x)$ A double headed arrow of length 1.



Physical Interpretations

$\delta(t)$ is a **strong** kick forward, 'unit impulse' at $t = 0$. The area under it is 1. You can calculate that the effect of a force $\delta(t)$ when applied to a particle is to produce an instantaneous change in its velocity.

$\delta'(t)$ is a **very strong** kick forwards followed immediately by an equal strong kick backward at $t = 0$. The area under each spike is $+\infty$ for every non-zero ϵ. We will see that the effect of the force $\delta''(t)$ when applied to a particle is to produce an instantaneous change in its position with no net change of velocity.

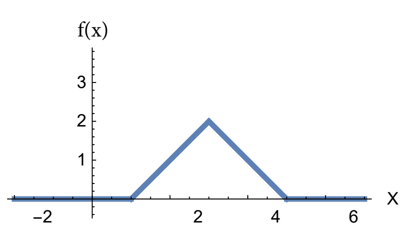
Note The Dirac Delta function is a genuine hyperreal based function, not the hyperreal extension of a real function. Clearly we need a hyper-hyperreal calculus based on hyperinfinitesimals $\mathbf{dx}$ smaller in size than any positive infinitesimal, in particular the ϵ we used in the pre-delta function. However, we can still get by thinking of the derivative as a slope and the integral as an area: there is no need here to develop a full hyper-hyperreal calculus.

Exercises

Work some from 1 to 9. Marvel the advanced application appendix.

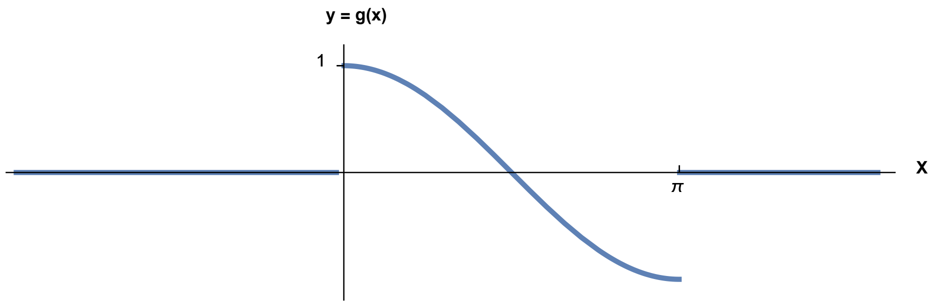
1. A. Write a formula for $f(x)$ shown below in terms of H analytically. Find $f(x)$ in terms of H and δ and simplify using the Sifting Property. Finally verify analytically that the Fundamental Theorem of Calculus holds.

B. Repeat part A but this time do graphically.

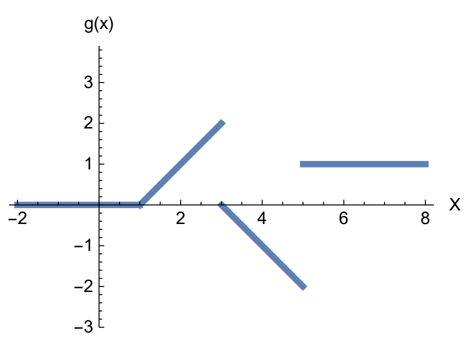


C. Why doesn't $f'(x)$ involve a delta function?

2. Repeat parts 1 A and 1 B for the function below.



3. Repeat parts 1 A and 1 B for the function below.



4. Verify each by graphing and/or analytically.
   a. $\int_{-\infty}^{x} H(t)\,dt = x\,H(x)$
   b. $\int_{-\infty}^{x}\int_{-\infty}^{t} H(s)\,ds\,dt = \frac{1}{2}x^2\,H(x)$

5. Verify each Sifting Property of $\delta\,'(x)$.
   a. $f(x)\,\delta\,'(x) = -f\,'(0)\delta(x)$
   b. $f(x)\,\delta\,'(x-a) = -f\,'(a)\delta(x-a)$

6. Graph each.
   a. $\delta(x)$
   b. $\int_{-\infty}^{x} \delta(t)\,dt$ Write a formula for this function.
   c. $\int_{-\infty}^{x}\int_{-\infty}^{t} \delta(s)\,ds\,dt$ Write a formula for this function.

7. Criticise the graphical representation of $\delta\,'(x)$.

8. a. Draw an approximation $\delta_\epsilon'(x)$ for $\delta\,''(x)$.
   b. How would you show the graph of $\delta\,''(x)$?
   c. What is the Sifting Property of $\delta\,''(x)$?

9. Show that $\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$.

10. A particle with mass 1 initially at rest at $x = 0$. At $t = 0$, the particle is subject to the following forces:
    a. $F = H(t)$
    b. $F = \delta(t)$
    c. $F = \delta\,'(t)$.

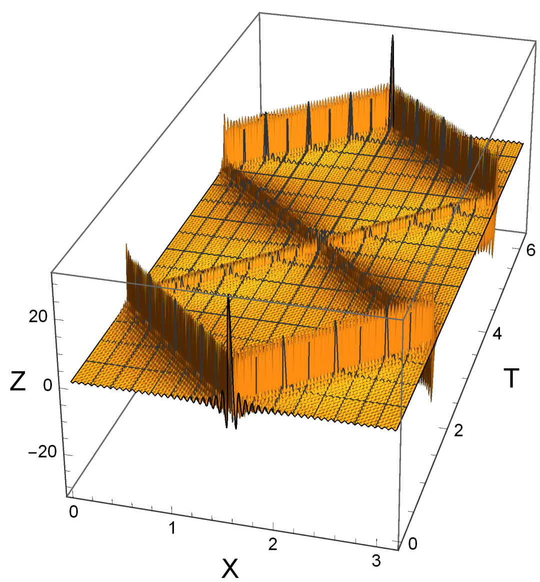
Use Newton's Law, $F = m\,a$, to find the velocity and position as a function of time in each case. Graph

11. a. Find the antiderivatives of the functions f(x) and g(x) in exercises 1 and 2.
Notation: $F(x) = \int_{-\infty}^{x} f(t)\,dt$. Verify the ***Fundamental Theorem of Calculus*** for these functions.

Future types of applications, FYI.

I. Application to a vibrating string

Find the solution for the plucked string problem with $f(x) = \delta(x - \pi/2)$. How would you produce this initial condition? The solution graphed below is the solution of the wave equation, $\partial^2 z/\partial x^2 = \partial^2 z/\partial t^2$. Study the solution and describe what happens.



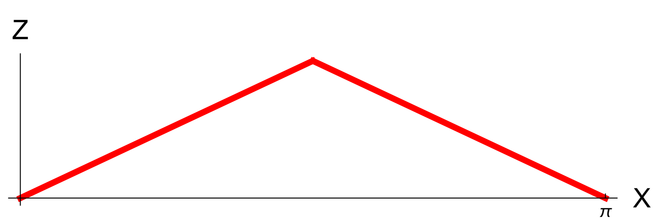
A string is fixed at $x = 0$ and $x = \pi$. It is drawn up at $\pi/2$ between two fingers into a pulse approximating a delta function and released at $t = 0$. The motion of the pulse between $t = 0$ and $t = 2\pi$ is shown.

II. Generalized Differentiability at a ridge - generalized derivatives work!

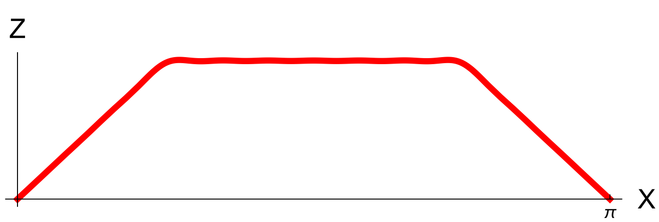
The solution for the plucked string problem with $f(x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi \end{cases}$.

Want to show that at a point where the solution is not differentiable in the ordinary sense, the solution satisfies the PDE in the generalized sense.

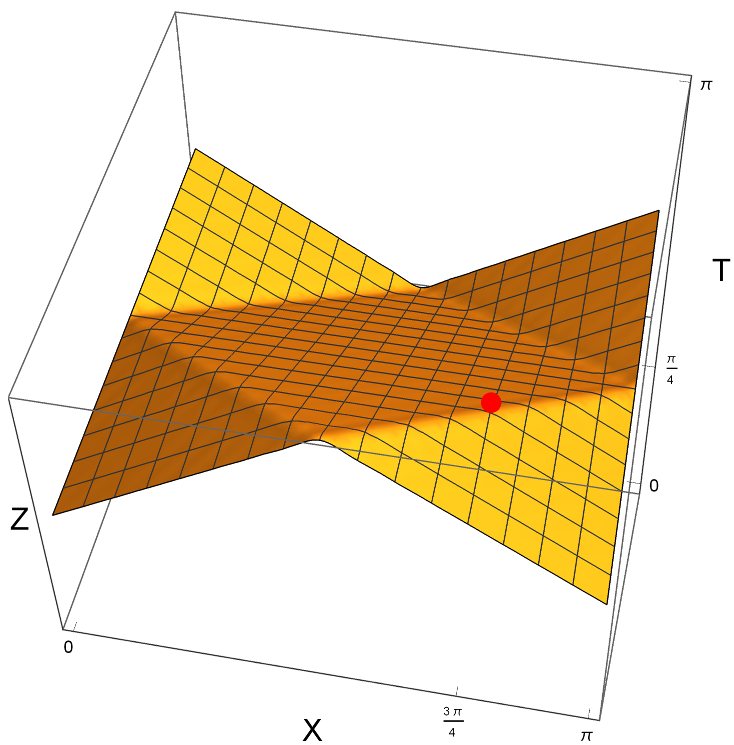
The string is plucked as shown below at $t = 0$.



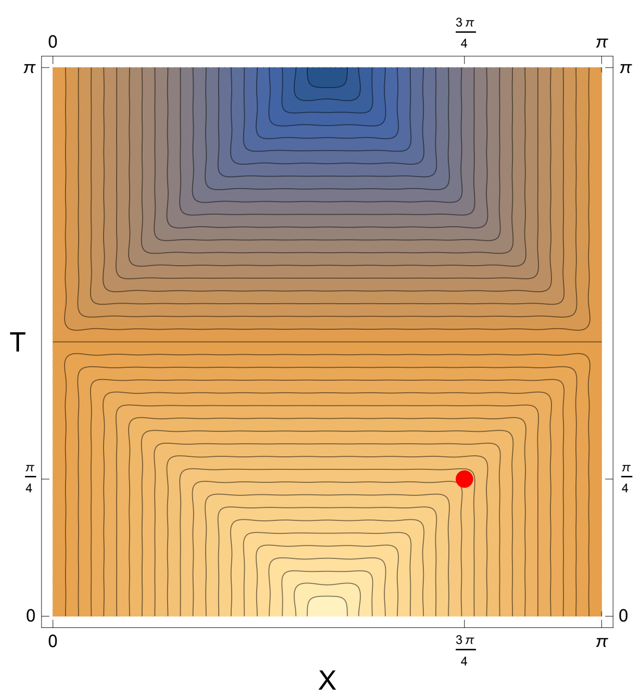
At $t = \frac{\pi}{4}$ it looks like this. Surprise!



One half of a period of vibration is shown below.

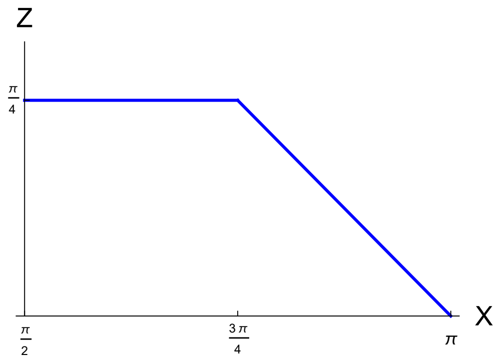


A contour plot of the above is drawn below.



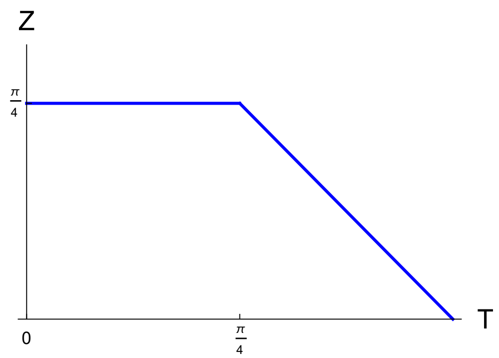
We will examine the solution at the red dot: $x = \frac{3\pi}{4}$, $t = \frac{\pi}{4}$.

The curve through the red point in the X-direction is



In elementary calculus , this solution does not have either a first or second derivative in either the X-direction or the T-direction at the corner point. Highly unsatisfactory (because the motion is real).

The line through the red point in the T-direction is



Exercise Draw the first and second derivatives of the graphs on the left. Of course they are equal respectively! Thus the wave equation is satisfied at the red point.

a. Graphically find the generalized derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial t}$. Note that they exist and are equal. Note: the symbol ∂ is used instead of d when there is more than one independent variable

b. Graphically find the generalized derivatives $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial t^2}$. Note that they exist and are equal. This means the wave equation at a sharp corner has a solution when generalized calculus is used, but not with ordinary calculus.

In generalized calculus , this solution does have both a first or second derivative in both the X-direction and the T-direction everywhere. Completely satisfactory.

Chapter 10 First 0rder Differential Equations

10.1 First Order Separable Differential Equations

The laws of growth of a quantity - particularly in finance, the natural sciences and engineering - are often expressed by equations where derivatives of the quantity appear. In this chapter we will begin the study of simple first order and second order differential equations.

First order differential equations have the forms

$$\frac{dy}{dx} = f(x, y) \qquad \text{or} \qquad F(x, y, \frac{dy}{dx}) = 0.$$

Their ***solutions*** have one arbitrary constant because in one way or another an indefinite integration is involved. The simple differential equation

$\frac{dy}{dx} = x$ has the ***general solution*** $y = \frac{x^2}{2} + C$.

The constant C is found by requiring the solution to pass through a point. If the solution goes through $(2, 2)$, then $C = 0$ and $y = \frac{x^2}{2}$.

How does a differential equation determine a solution? Let us look at an example.

Example Consider the differential equation $\frac{dy}{dx} = x + y$. Let us find the solution passing through the point $(0, 2)$.

Write the differential equation in its approximate differential form:

$$\Delta y \doteq (x + y)\Delta x.$$

Start at **(0, 1)**. Chose $\Delta x = 1$ (very large!). Find Δy. Get the point $(0+\Delta x, 1+\Delta y)$. Repeat.

(0, 1) $\Longrightarrow$ $\Delta x = 1, \Delta y = (0 + 1)\cdot 1$ $\Longrightarrow$ $(1, 2)$

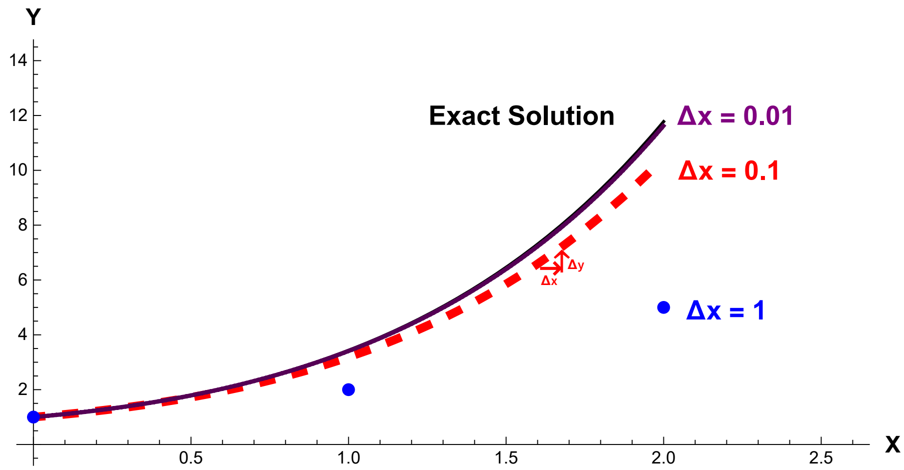
(1, 2) $\Longrightarrow$ $\Delta x = 1, \Delta y = (1 + 2)\cdot 1$ $\Longrightarrow$ $(2, 5)$

(2, 5) and so on. See the blue data points. Not bad? It goes roughly in the right direction when compared to the exact solution, the black curve.

For a better solution take $\Delta x = 0.1$, say. Then you would get the red data. Much better. You could do this by hand in a few minutes with good mental arithmetic and concentration

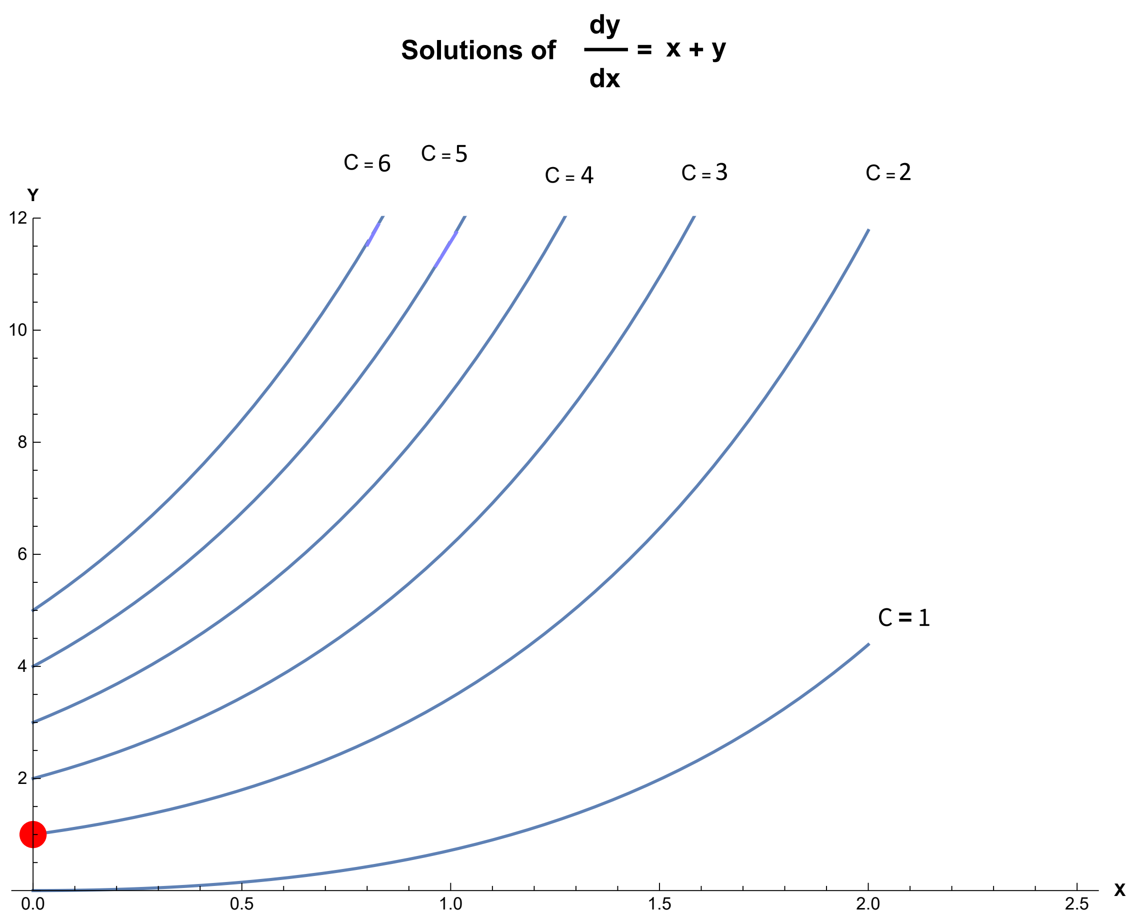
For a very good solution take $\Delta x = 0.01$. Then you would get the purple data. The data points barely peek out from the under exact solution on the right. You could do this with some simple programming.

For a solution asymptotically equal to the exact solution, take $\Delta x = dx$, an infinitesimal, of course! This is how a first order differential determines a solution.



The exact general solution of $\frac{dy}{dx} = x + y$ is $y = Ce^x - x - 1$. It is graphed for several values of C below. It is a homework problem to find this solution. The solution passing through $(0, 1)$ is

$$y = 2e^x - x - 1.$$



First order differential equations, Variables Separable

Both mathematicians and students like exact analytic methods which work for a large number of differential equations. One type is the variables separable equations:

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx \qquad \text{separating variables}$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx \qquad \text{integrating}$$

The solution.

Example Find the general solution of $\frac{dy}{dx} = \frac{y}{x}$.

$\frac{dy}{y} = \frac{dx}{x}$ separating variables

$\int \frac{dy}{y} = \int \frac{dx}{x}$ integrating

$\ln y = \ln x + \ln C$ equivalent to C as the constant of integration. Why?

$\ln y - \ln x = \ln C$ Property 1 of logs

$\ln(y/x) = \ln C$ Property 2 of logs

$y = Cx$ Exponentiation

Example Find the solution of the initial value problem $\begin{cases} \frac{dy}{dx} = \frac{x}{y} \\ y(0) = 1 \end{cases}$.

$\frac{dy}{dx} = \frac{x}{y}$

$y\,dy = x\,dx$ separating variables

$\int y\,dy = \int x\,dx$ integrating

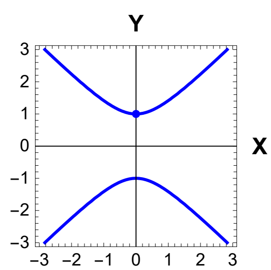
$\frac{y^2}{2} = \frac{x^2}{2} + \frac{C^2}{2}$ looking ahead

$y^2 - x^2 = C^2$ a hyperbola

$1^2 - 0^2 = C^2$ initial condition

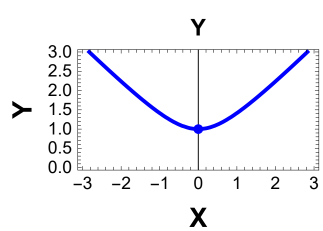
$C^2 = 1$

Solution: $\mathbf{y^2 - x^2 = 1}$?



Since the bottom curve does not pass through the point $(0, 1)$, the correct solution is:

Solution: $\mathbf{y = \sqrt{1 + x^2}}$



Sometimes separation of variables is not entirely obvious.

Example Find the general solution of $\frac{dy}{dx} = e^{x-y}$.

$$\frac{dy}{dx} = e^{x-y}$$

$$= e^x e^{-y} \qquad \text{property of exponents}$$

$$e^y dy = e^x dx \qquad \text{separating variables}$$

$$\int e^y\, dy = \int e^x\, dx \quad \text{integrating}$$

$$e^y = e^x + C$$

$$y = \ln(e^x + C)$$

Note: $y = \ln e^x + \ln C$
$= x + D$
is wrong. Why?

Exercise 10.1.0 Find the exact solution of $\frac{dy}{dx} = x + y$. Hint: make the change of variable $u = x + y$.

Exercise 10.1.1 *Which of the following equations are separable?*

(a) $y' = \sin(ty)$

(b) $y' = e^t e^y$

(c) $y y' = t$

(d) $y' = (t^3 - t) \arcsin(y)$

(e) $y' = t^2 \ln y + 4t^3 \ln y$

Exercise 10.1.2 *Solve* $y' = 1/(1+t^2)$.

Exercise 10.1.3 *Solve the initial value problem* $y' = t^n$ *with* $y(0) = 1$ *and* $n \geq 0$.

Exercise 10.1.4 *Solve* $y' = \ln t$.

Exercise 10.1.5 *Identify the constant solutions (if any) of* $y' = t \sin y$.

Exercise 10.1.6 *Identify the constant solutions (if any) of* $y' = te^y$.

Exercise 10.1.7 *Solve* $y' = t/y$.

Exercise 10.1.8 *Solve* $y' = y^2 - 1$.

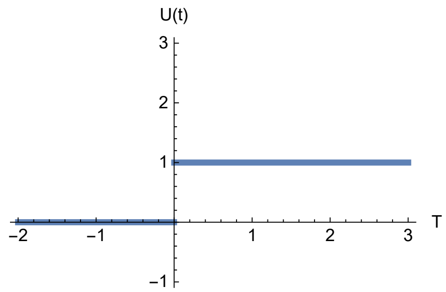
Exercise 10.1.9 *Solve* $y' = t/(y^3 - 5)$. *You may leave your solution in implicit form: that is, you may stop once you have done the integration, without solving for* y.

Exercise 10.1.10 *Find a non-constant solution of the initial value problem* $y' = y^{1/3}$, $y(0) = 0$, *using separation of variables. Note that the constant function* $y(t) = 0$ *also solves the initial value problem. This shows that an initial value problem can have more than one solution.*

10.6 Generalized Functions Analytically

Review The Heaviside Function H(t)

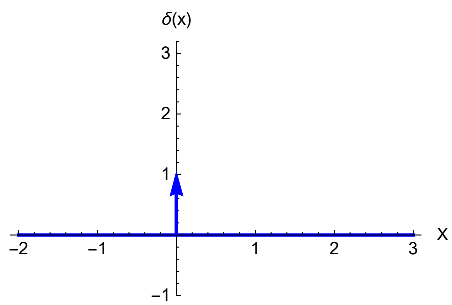
$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



Review The Dirac Delta Function δ(x):

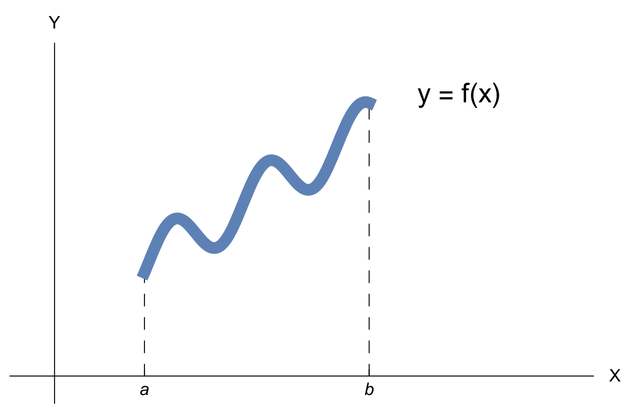
1. $\delta(x) = 0,\ x \neq 0$
2. $\int_{-\infty}^{x} \delta(t)\,dt = H(x)$ or conversely $\frac{d}{dx}H(x) = \delta(x)$

Graphically $\delta(x)$ is shown by an upward unit arrow with its tail at $x = 0$ on the X-axis.



Using the Heaviside Step function to write equations of generalized functions

First we do this for one function segment.

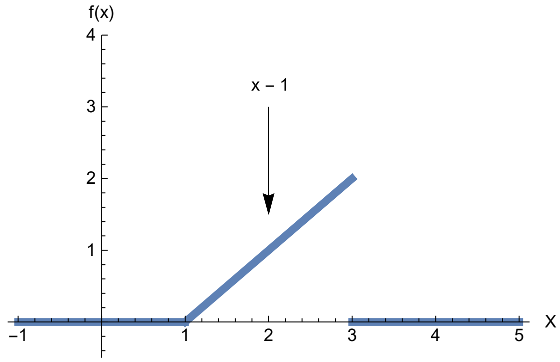


y = 'turn on f(x) at x = a' and then 'turn off f(x) at x = b'

$= f(x)H(x-a) - f(x)H(x-b)$.

For general generalized functions apply the above technique to each function segment.

Example*



Its equation is

f(x) = (x - 1)H(x-1) - (x - 1)H(x-3).

Generalized Calculus Properties of H(x-a) and δ(x-a)

1. δ(x) = 0, x ≠ 0	1. δ(x-a) = 0, x ≠ a
2. $\int_{-\infty}^{x} \delta(t)\, dt$ = H(x)	2. $\int_{-\infty}^{x} \delta(t-a)\, dt$ = H(x-a)
3. H '(x) = δ(x)	3. H '(x-a) = δ(x-a)
4. f(x) δ(x) = f(0) δ(x)	4. f(x) δ(x-a) = f(a) δ(x-a) 'Sifting Property'

Proofs The only mystery is #4, the Sifting Property. f(x) δ(x) = f(0) δ(x) follows from #1, since the only value of f that 'counts' is at x = 0. The Sifting Property is useful in simplifying expressions involving the delta function.

Example Let us look at the previous example.

f(x) = (x - 1)H(x-1) - (x - 1)H(x-3).

By the Product Rule

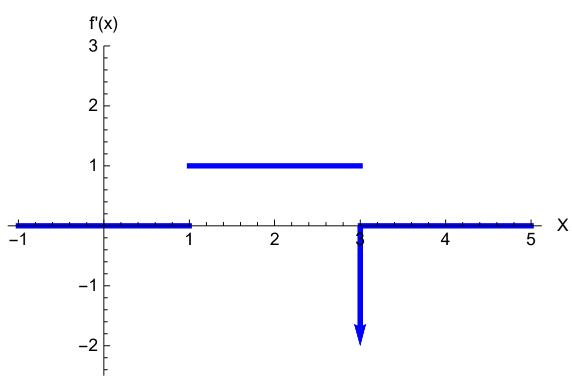
f '(x) = 1 H(x-1) + (x - 1)δ(x-1) - 1 H(x-3) - (x - 1)δ(x-3)

= H(x-1) - H(x-3) - 2δ(x-3)   since by the Sifting Property:

(x - 1)δ(x-1) = (1 - 1)δ(x-1) = 0

(x - 1)δ(x-3) = (3 - 1)δ(x-3) = 2δ(x-3)

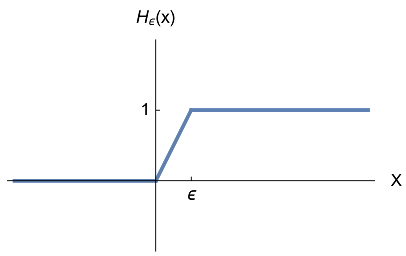
Note that this derivative does not involve a delta function at x = 1. This is because f is continuous there. The graph of the derivative is

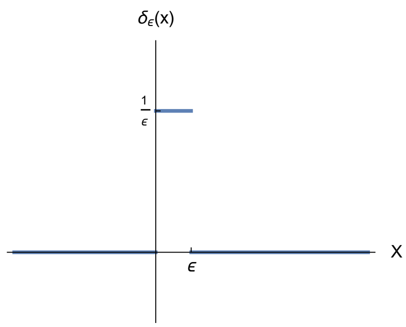


which looks just right.

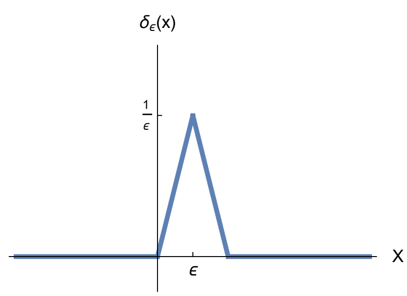
Interpretation of δ '(x) The delta function and its derivative are important in applications. Their geometric approximations are useful in understanding their properties.

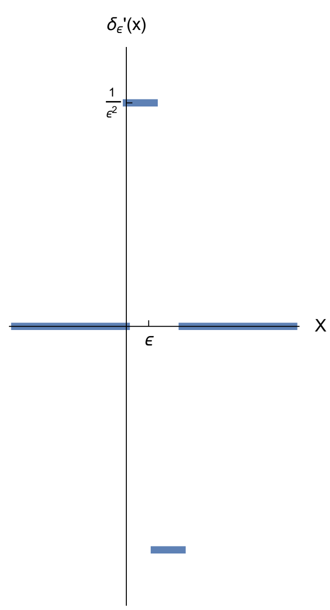
Here are the approximations of H(x), δ(x), and δ '(x). You can verify them by starting at the top with differentiation **or** starting at the bottom with integration.



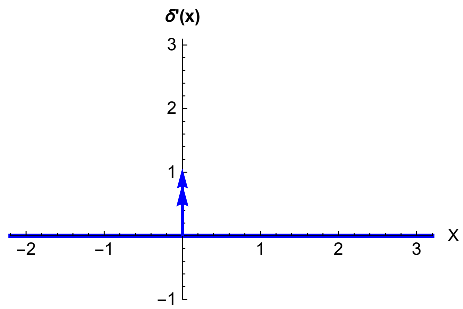


To find the derivative of $\delta_\epsilon(x)$ we will use another version of it in triangular form which also has area 1.





Graph of $\delta'(x)$ A double headed arrow of length 1.



Physical Interpretations

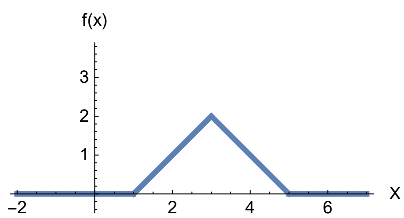
H(t) is a constant force of magnitude 1 starting at $t = 0$.

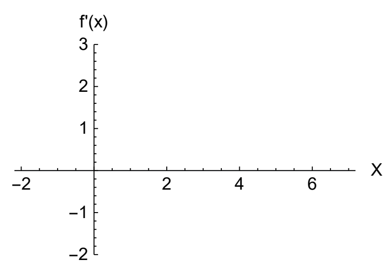
δ(t) is a **strong** kick forward, 'unit impulse' at $t = 0$. The area under it is 1. You can calculate that the effect of a force δ(t) when applied to a particle is to produce an instantaneous change in its velocity.

δ'(t) is a **very strong** kick forwards followed immediately by an equal strong kick backward at $t = 0$. The area under each spike is $+\infty$ as $\epsilon \to 0$. We will see that the effect of the force δ'(t) when applied to a particle is to produce an instantaneous change in its position with no net change of velocity.

Exercises Work 1 to 6, 10, 11. Marvel at advanced applications.

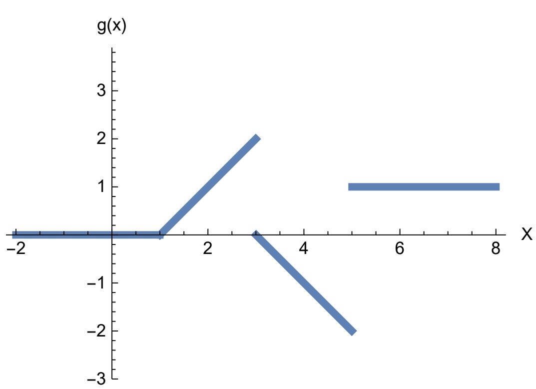
1. Write a formula for f(x) in terms of H. Find f '(x) and simplify. Graph f '(x) by hand and computer.

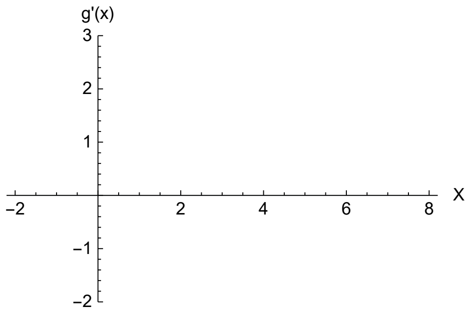




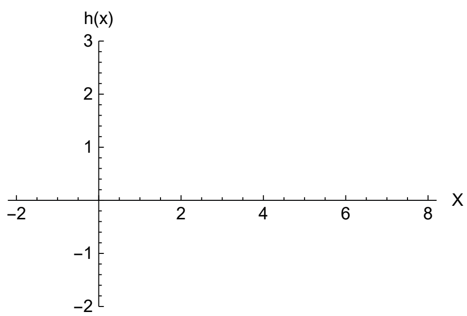
Why doesn't f '(x) involve a delta function? Do both graphically and analytically.

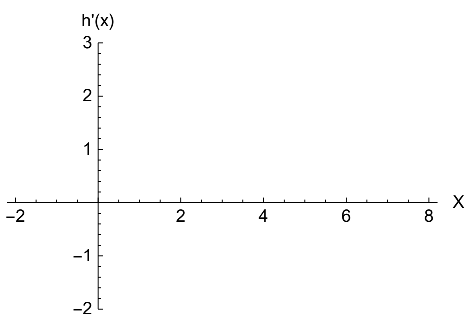
2. Write a formula for g(x) in terms of U. Find g '(x) and simplify. Graph g '(x) by hand and computer.





3. Invent your own h(x). Write a formula for h(x) in terms of U. Find h '(x) and simplify. Graph h(x) and h '(x) by hand and by computer. h should have two non-zero segments one of which is non-constant.





4. Verify each by graphing.
   a. $\int_{-\infty}^{x} H(t)\,\mathrm{dt} = \mathrm{x\,H(x)}$
   b. $\int_{-\infty}^{x}\int_{-\infty}^{t} H(s)\,\mathrm{ds\,dt} = \frac{1}{2}x^2\,\mathrm{H(x)}$

5. Verify each Sifting Property of δ '(x).
   a. f(x) δ '(x) = -f '(0)δ(x)
   b. f(x) δ '(x-a)= -f '(a)δ(x-a)

6. Graph each.
   a. δ(x)
   b. $\int_{-\infty}^{x} \delta(t)\,\mathrm{dt}$ Write a formula for this function.
   c. $\int_{-\infty}^{x}\int_{-\infty}^{t} \delta(s)\,\mathrm{ds\,dt}$ Write a formula for this function.

7. Criticise the graphical representation of δ ' (x).

8. a. Draw an approximation $\delta_\epsilon'(x)$ for $\delta''(x)$.
   b. How would you show the graph of $\delta''(x)$?
   c. What is the Sifting Property of $\delta''(x)$?

9. Show that $\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$.

10. A particle with mass 1 initially at rest at $x = 0$. At $t = 0$, the particle is subject to the following forces:
    a. $F = H(t)$
    b. $F = \delta(t)$
    c. $F = \delta'(t)$.

Use Newton's Law, $F = m\, a$, to find the velocity and position as a function of time in each case.

11. Show that f(x) in Example * satisfies the fundamental theorem of calculus

$$\int_{-\infty}^{x} f'(t)\, dt = f(x).$$

Solutions 10.1

10.1.2 $y = \arctan t + C$

10.1.3 $y = \dfrac{t^{n+1}}{n+1} + 1$

10.1.4 $y = t\ln t - t + C$

10.1.5 $y = n\pi$, for any integer n.

10.1.6 none

10.1.7 $y = \pm\sqrt{t^2 + C}$

10.1.8 $y = \pm 1$, $y = (1 + Ae^{2t})/(1 - Ae^{2t})$

10.1.9 $y^4/4 - 5y = t^2/2 + C$

10.1.10 $y = (2t/3)^{3/2}$

10.2 First Order Homogeneous Linear Equations

A simple, but important and useful, type of separable equation is the **first order homogeneous linear equation**:

Definition 10.1.2.1: First Order Homogeneous Linear Equation

A first order homogeneous linear differential equation is one of the form $y' + p(t)y = 0$ or equivalently $y' = -p(t)y$.

"Homogeneous" refers to the zero on the right side of the equation, provided that y' and y are on the left. "Linear" in this definition indicates that both y' and y appear independently and explicitly; we don't see y' or y to any power greater than 1, or multiplied by each other (i.e. $y'y$).

Example 10.2.1: Linear Examples

The equation $y' = 2t(25-y)$ *can be written* $y' + 2ty = 50t$. *This is linear, but not homogeneous. The equation* $y' = ky$, *or* $y' - ky = 0$ *is linear and homogeneous, with a particularly simple* $p(t) = -k$. *The equation* $y' + y^2 = 0$ *is homogeneous, but not linear.*

Since first order homogeneous linear equations are separable, we can solve them in the usual way:

$$\begin{aligned} \frac{\mathrm{dy}}{\mathrm{dx}} &= -p(t)y \\ \int \frac{1}{y}\,dy &= \int -p(t)\,dt \\ \ln|y| &= P(t) + \quad C \\ y &= \pm e^{P(t)+\mathrm{C}} \\ y &= Ae^{(P\ t)}, \qquad \text{where} \quad \pm \mathrm{e}^{\mathrm{C}} = \mathrm{A} \end{aligned}$$

where $P(t)$ is an anti-derivative of $-p(t)$. As in previous examples, if we allow $A = 0$ we get the constant solution $y = 0$.

Example 10.2.2 Solving an IVP

Solve the initial value problem

$$y' + y\cos t = 0,$$

subject to $y(0) = 1/2$ *and* $y(2) = 1/2$.

Solution. We start with

$$P(t) = \int -\cos t\,dt = -\sin t,$$

so the general solution to the differential equation is

$$y = Ae^{-\sin t}.$$

To compute A we substitute:

$$\frac{1}{2} = Ae^{-\sin 0} = A,$$

so the solutions is

$$y = \frac{1}{2}e^{-\sin t}.$$

For the second problem,

$$\begin{aligned} \frac{1}{2} &= Ae^{-\sin 2} \\ A &= \frac{1}{2}e^{\sin 2} \end{aligned}$$

so the solution is

$$y = \frac{1}{2}e^{\sin 2}e^{-\sin t}.$$

Example 10.2.2

Solve the initial value problem $ty' + 3y = 0$, $y(1) = 2$, *assuming* $t > 0$.

Solution. We write the equation in standard form: $y' + 3y/t = 0$. Then

$$P(t) = \int -\frac{3}{t}\,dt = -3\ln t$$

and

$$y = Ae^{-3\ln t} = At^{-3}.$$

Substituting to find A: $2 = A(1)^{-3} = A$, so the solution is

$$y = 2t^{-3}.$$

Exercises for 10.2

Find the general solution of each equation in the following exercises.

Exercise 10.2.1 $y' + 5y = 0$

Exercise 10.2.2 $y' - 2y = 0$

Exercise 10.2.3 $y' + \dfrac{y}{1+t^2} = 0$

Exercise 10.2.4 $y' + t^2 y = 0$

In the following exercises, solve the initial value problem.

Exercise 10.2.5 $y' + y = 0$, $y(0) = 4$

Exercise 10.2.6 $y' - 3y = 0$, $y(1) = -2$

Exercise 10.2.7 $y' + y\sin t = 0$, $y(\pi) = 1$

Exercise 10.2.8 $y' + ye^t = 0$, $y(0) = e$

Exercise 10.2.9 $y' + y\sqrt{1+t^4} = 0$, $y(0) = 0$

Exercise 10.2.10 $y' + y\cos(e^t) = 0$, $y(0) = 0$

Exercise 10.2.11 $ty' - 2y = 0$, $y(1) = 4$

Exercise 10.2.12 $t^2 y' + y = 0$, $y(1) = -2$, $t > 0$

Exercise 10.2.13 $t^3 y' = 2y$, $y(1) = 1$, $t > 0$

Exercise 10.2.14 $t^3 y' = 2y$, $y(1) = 0$, $t > 0$

Exercise 10.2.15 *A function* $y(t)$ *is a solution of* $y' + ky = 0$. *Suppose that* $y(0) = 100$ *and* $y(2) = 4$. *Find* k *and find* $y(t)$.

Exercise 10.2.16 *A function* $y(t)$ *is a solution of* $y' + t^k y = 0$. *Suppose that* $y(0) = 1$ *and* $y(1) = e^{-13}$. *Find* k *and find* $y(t)$.

Exercise 10.2.17 *A bacterial culture grows at a rate proportional to its population. If the population is one million at* $t = 0$ *and 1.5 million at* $t = 1$ *hour, find the population as a function of time.*

Exercise 10.2.18 *A radioactive element decays with a half-life of 6 years. If a mass of the element weighs ten pounds at* $t = 0$, *find the amount of the element at time* t.

Solutions 10.2

10.2.1 $y = Ae^{-5t}$

10.2.2 $y = Ae^{2t}$

10.2.3 $y = Ae^{-\arctan t}$

10.2.4 $y = Ae^{-t^3/3}$

10.2.5 $y = 4e^{-t}$

10.2.6 $y = -2e^{3t-3}$

10.2.7 $y = e^{1+\cos t}$

10.2.8 $y = e^2 e^{-e^t}$

10.2.9 $y = 0$

10.2.10 $y = 0$

10.2.11 $y = 4t^2$

10.2.12 $y = -2e^{(1/t)-1}$

10.2.13 $y = e^{1-t^{-2}}$

10.2.14 $y = 0$

10.2.15 $k = \ln 5$, $y = 100e^{-t \ln 5}$

10.2.16 $k = -12/13$, $y = \exp(-13t^{1/13})$

10.2.17 $y = 10^6 e^{t \ln(3/2)}$

10.2.18 $y = 10e^{-t \ln(2)/6}$

10.3 First Order Linear Equations

$$y' + p(t)y = f(t)$$

A common method for solving such a differential equation is by multiplying both sides by the **integrating factor**:

$$e^{P(t)}$$

where P(t) is an antiderivative of p(t).

Then

$$e^{P(t)}y' + e^{P(t)}p(t)y = e^{P(t)}f(t)$$

$$\frac{d}{dt}(e^{P(t)}y) = e^{P(t)}f(t). \qquad \text{Product Rule}$$

Integrating both sides gives

$$e^{P(t)}y = \int e^{P(t)}f(t)\,dt$$

$$y = e^{-P(t)}\int e^{P(t)}f(t)\,dt + C, \ \textbf{ the solution}$$

Example Solve $\frac{dy}{dt} - 2y = 6$

Integrating Factor = $e^{\int -2\,dt} = e^{-2t}$

$\frac{dy}{dt}e^{-2t} - 2ye^{-2t} = 6e^{-2t}$

$\frac{d}{dt}[ye^{-2t}] = 6e^{-2t}$

$ye^{-2t} = -3e^{-2t} + C$ y integrating both sides

y = $-3 + Ce^{2t}$

Note: in the exercises, use the procedure of the example. Don't use the red formula.

Exercises for 10.3

NOTE Big people, when no one is watching, use Wolfram Alpha or other resources to evaluate difficult integrals.

In the following exercises, find the general solution of the equation.

Exercise 10.3.1 $y' + 4y = 8$

Exercise 10.3.2 $y' - 2y = 6$

Exercise 10.3.3 $y' + ty = 5t$

Exercise 10.3.4 $y' + e^t y = -2e^t$

Exercise 10.3.5 $y' - y = t^2$

Exercise 10.3.6 $2y' + y = t$

Exercise 10.3.7 $ty' - 2y = 1/t,\ t > 0$

Exercise 10.3.8 $ty' + y = \sqrt{t},\ t > 0$

Exercise 10.3.9 $y' \cos t + y \sin t = 1,\ -\pi/2 < t < \pi/2$

Exercise 10.3.10 $y' + y \sec t = \tan t,\ -\pi/2 < t < \pi/2$

10.3.11 a. Derive again the general solution of

$$\begin{cases} y' + p(t)y = 0 \\ y(0) = y_o. \end{cases}$$

b. Show that the above system can be written in the one-line form preferred in some applications by

$$y' + p(t)y = y_o\,\delta(t).$$

Important Note

Show it is correct by solving it. When you integrate, use generalized integration and the Sifting Property.

The idea here is that all solutions are 0 at minus infinity and get a displacement at t = 0 because of an impulse force.

Solutions 10.3

10.3.1 $y = Ae^{-4t} + 2$

10.3.2 $y = Ae^{2t} - 3$

10.3.3 $y = Ae^{-(1/2)t^2} + 5$

10.3.4 $y = Ae^{-e^t} - 2$

10.3.5 $y = Ae^t - t^2 - 2t - 2$

10.3.6 $y = Ae^{-t/2} + t - 2$

10.3.7 $y = At^2 - \dfrac{1}{3t}$

10.3.8 $y = \dfrac{c}{t} + \dfrac{2}{3}\sqrt{t}$

10.3.9 $y = A\cos t + \sin t$

10.3.10 $y = \dfrac{A}{\sec t + \tan t} + 1 - \dfrac{t}{\sec t + \tan t}$

Memory work in 1958: all the formulas

Derivative Formulas

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

$\frac{d}{dx}(e^x) = e^x$ $\qquad$ $\frac{d}{dx}(a^x) = a^x \ln a$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ $\qquad$ $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

$\frac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\qquad$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\qquad$ $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\qquad$ $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ $\qquad$ $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\sinh x) = \cosh x$ $\qquad$ $\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$ $\qquad$ $\frac{d}{dx}(\coth x) = -\text{csch}^2 x$

$\frac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x \tanh x$ $\qquad$ $\frac{d}{dx}(\text{csch}\, x) = -\text{csch}\, x \coth x$

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$ $\qquad$ $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1$ $\qquad$ $\frac{d}{dx}(\coth^{-1} x) = \frac{-1}{1-x^2}, |x| > 1$

$\frac{d}{dx}(\text{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$ $\qquad$ $\frac{d}{dx}(\text{csch}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

The above list includes derivatives of all the basic elementary functions.

Integral Formulas

$\int u^n du = u^{n+1} + C$ $\qquad$ $\int \frac{du}{u} = \ln|u| + C$

$\int e^u du = e^u + C$ $\qquad$ $\int a^u du = \frac{a^u}{\ln a} + C$

$\int \cos u\, du = \sin u + C$ $\qquad$ $\int \sin u\, du = -\cos u + C$

$\int \sec^2 u\, du = \tan u + C$ $\qquad$ $\int \csc^2 u\, du = -\cot u + C$

$\int \sec u \tan u\, du = \sec u + C$ $\qquad$ $\int \csc u \cot u\, du = -\csc u + C$

$\int \tan u\, du = \ln|\sec u| + C$ $\qquad$ $\int \cot u\, du = \ln|\sin u| + C$

$\int \sec u\, du = \ln|\sec u + \tan u| + C$ $\qquad$ $\int \csc u\, du = \ln|\csc u - \cot u| + C$

$\int \cosh u\, du = \sinh u + C$ $\qquad$ $\int \sinh u\, du = \cosh u + C$

$\int \text{sech}^2 u\, du = \tanh u + C$ $\qquad$ $\int \text{csch}^2 u\, du = -\coth u + C$

$\int \text{sech}\, u \tanh u\, du = -\text{sech}\, u + C$ $\qquad$ $\int \text{csch}\, u \coth u\, du = -\text{csch}\, u + C$

$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$ $\qquad$ $\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$

$\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + C$ $\qquad$ $\int \frac{du}{1+u^2} = \tan^{-1} u + C$

$\int \frac{du}{1-u^2} = \tanh^{-1} u + C, |u| < 1$ $\qquad$ $\int \frac{du}{1-u^2} = \coth^{-1} u + C, |u| > 1$

$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C$ $\qquad$ $\int \frac{du}{u\sqrt{1-u^2}} = \text{sech}^{-1} u + C$

$\int \frac{du}{|u|\sqrt{u^2+1}} = \text{csch}^{-1} u + C$

Integrals of basic inverse functions can always be done by integration by parts.

Memory work in 2015: the boldface formulas