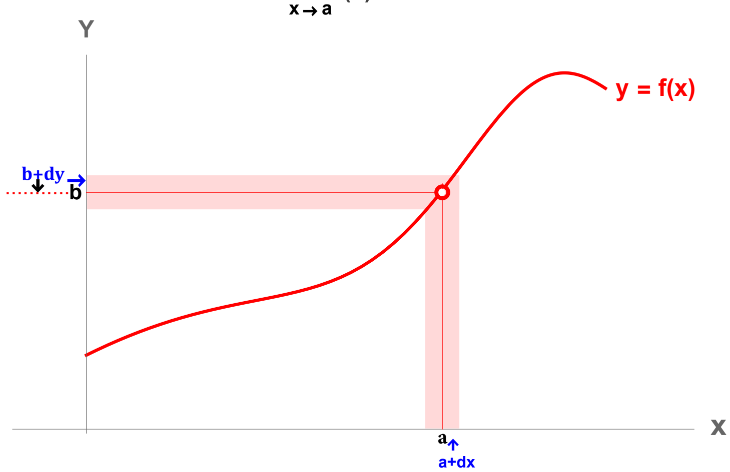
THE SITUATION. Let dx be any non-zero infinitesimal.

f(a) is an indeterminate form, undefined.
Compute f(a+dx) = b+dy instead.
Set dy = 0.
b is the number we need for calculus.

We write $\lim_{x \to a} f(x) = b$



Historical Problem First we require dx ≠ 0 along the X-axis.
Then we set dy = 0 by taking dx = 0.
How can dx be both zero and non-zero?
No problem. They occur under different circumstances.

If you can't find it under the rock,
look near the rock.

APEX CALCULUS I

Version 4.0

Gregory Hartman, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Contributing Authors

Troy Siemers, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Brian Heinold, Ph.D.

Department of Mathematics and Computer Science

Mount Saint Mary's University

Dimplekumar Chalishajar, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Editor

Jennifer Bowen, Ph.D.

Department of Mathematics and Computer Science

The College of Wooster

Copyright © 2018 Gregory Hartman
Licensed to the public under Creative Commons Attribution-Noncommercial 4.0 International Public License

Apex Infinitesimal Calculus, Volume I

Revised Edition 1.03

William Freed
Concordia University Of Edmonton

Click on red box for desired section

Apex Infinitesimal Calculus, Volume I

Preface

Why an infinitesimal calculus approach? For many years I used textbooks primarily as a source of problem sets but did most the theory and derivations using infinitesimal analysis. It went well.

What is wrong with with the ϵ-δ limit approach?

Students did not master solving absolute value & inequality statements before university (High School students tend to think they had a 'nice' mathematics teacher if they did not do much with word problems, piecewise defined functions or absolute value & inequality statements!)

ϵ-δ calculations are ugly and difficult!

Often ϵ-δ limits are treated very lightly or not at all and students do calculus with a minimal understanding of limits and get through calculus by memorizing formulas and mimicking textbook examples. **Some** important basic theorems such as the ***Extreme Value Theorem*** or the ***Riemann Integrability of a Continuous Function on a Closed Interval*** are simply too hard to do.

$\frac{dy}{dx}$ is not a fraction; The differentials dx and dy are not infinitesimals.

Few engineers or scientists use the traditional limit approach in their work, either in their university courses or in their work life. Calculus courses should help these in applied science to use good style with a clear understanding rather than have it hinder them.

What is good about the infinitesimal approach?

Calculations tend to look like standard Algebra 10 calculations. (Grade 10 algebra is often the last algebra they have really understood and mastered!)

Proofs of the *Extreme Value Theorem* and the **Riemann Integrability of a Continuous on a Closed Interval** are easy and intuitive and could be understood by a calculus student with only a successful grade 10 algebra background.

We do prove all the hard theorems in class; no one every gave me a raspberry for doing that.

$\frac{dy}{dx}$ is a fraction. Writing $\frac{dy}{dx} = f\,'(x)$ is only infinitesimally wrong!

There is a most excellent equivalence relation, asymptotic equality $\approx$, available for simplifying calculations; it encourages input by intuition; its misuse tends not to affect the final answer.

Did an engineering student ever use the Mean Value Theorem?

Also traditional applied notation is used throughout: $a < x < b$ not (a, b), $f(g(x))$ not $f \circ g(x)$.

Why did I make this patchwork calculus textbook?

I had hundreds of pages of infinitesimal based theory and applications handout sheets. I like the idea of Open Commons textbooks. Last summer our department was assigned a student partially subsidized by a government grant. No one had work for him. I thought,"Hey, we could make an infinitesimal calculus book." I informed our department chair. She wanted me to do it in LATEX. I thought of the effort involved. It would take about three years of drudgery to learn LATEX and type up a manuscript and do its many revisions. I worried; I'm 80 and would the textbook or the dementia win out? I decided we could do a "*3 Month Infinitesimal Calculus*" book to get started. The department chipped in for two copies of Acrobat Pro. Dallas McIntosh, my student assistant, was a great organizer and soon learned how to work around some of it's pdf rigidities.

You are invited to use this textbook, make suggestions, proofread or be an editor or . . . .

Chapter 0 Beginnings & Refreshments

A purpose of a university education is to produce experts in their major fields of study. Experts are required for teaching, doing original research, those who apply advanced knowledge to solving practical problems or those who are intellectually curious. Part of this expertise is understanding the background of their knowledge from its beginnings to your current level of study. That's why we start with the counting numbers. In this chapter we also do a basic algebra and function review. We finish with a new yet old kind of number system, one which includes very small numbers called infinitesimals, which allows us to study calculus at both an intuitive and at the same time at at a more advanced level.

THE PATH TO CALCULUS

Real Numbers
$\longrightarrow$ **Algebra**
$\longrightarrow$ **Functions**
$\longrightarrow$ **Continuity & Limits** } **via infinitesimals**
$\longrightarrow$ **Calculus!**
The Derivative
The Definite Integral
The Fundamental Theorem of Calculus

0.1 The Real Numbers

We begin with a short review of the real numbers and functions and graphing. Other topics in algebra will be reviewed as required.

Preliminaries

We start with a statement of a few logical symbols that we often will use as well as some properties of equals.

Logical Symbols For the sake of brevity we often use the following logical symbols for statements A and B.

$\Rightarrow$ "implies". $A \Rightarrow B$, or "A implies B" or "If A is true, then B is true"

$\Longleftrightarrow$ "means the same thing as" or "is equivalent to" or "if and only if".

Properties of Equals, =

$a = b$ means arithmetically that a and b are the same real numbers.

$a = b$ means geometrically that a and b are at the same place on the number line.

1. Reflexive property $a = a$
2. Symmetric property $a = b \Longleftrightarrow b = a$
3. Transitive property $a = b, b = c \Rightarrow a = c$

Mathematicians say that = is an *equivalence relation* because it has these three properties.

You may wish to check in this lesson that wherever we use = that these properties are consistent with its use.

Further properties of equals Useful when working with equations; in this context $\Longleftrightarrow$ means 'has the same solutions as'.

Addition Rule $a = b \Longleftrightarrow a + c = b + c$

Multiplication Rule $a = b \Longleftrightarrow a \cdot c = b \cdot c,\ c \neq 0$

These properties are true because, for example, in the addition rule of equals you could add $-c$ to both sides of the equation on the right and recover the equation on the left.

The Real Numbers

We will do an informal review of the real numbers. It is assumed you know rules of arithmetic of the real numbers and that elementary algebra is governed by to the same rules.

The set of *counting numbers* or *natural numbers* or *positive whole numbers* $\mathbb{N}$ is the set

1, 2, 3, 4, ⋯.

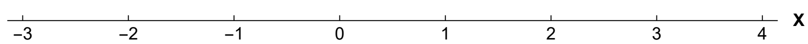
The natural numbers are useful for counting discrete objects such as jelly beans or kumquats. The sum or product of two natural numbers is a natural number. However, the difference may not be; $2 - 2$ and $3 - 5$ are not natural numbers. To allow for such subtractions we add 0 and the negative integers to the set of natural numbers to get a new set of numbers.

The set of *integers* $\mathbb{Z}$ contains the number 0 and the positive and negative whole numbers

⋯ −3, −2, −1, 0, 1, 2, 3, ⋯.

With integers it is possible to do all additions, subtractions and multiplications. However, the division of two integers is not necessarily an integer. To cure this problem we add fractions to the integers.

For the purposes of geometry and measurement it is convenient to place the integers equally spaced on a number line ordered from left to right.



The set of *rational numbers* $\mathbb{Q}$ (for Quotients) are ratios of integers $\frac{m}{n}$, $n \neq 0$.

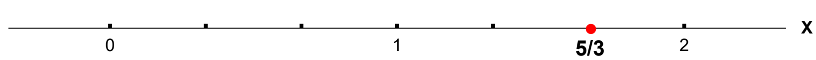
Two examples are

$$\frac{3}{4}$$

$$\frac{-13}{10} = \frac{13}{-10} = -\frac{13}{10} = -\frac{39}{30}$$

Integers are in the set of rational numbers with the understanding that $m \equiv \frac{m}{1}$; we say a ***rational extension*** of the integer m is the rational number $\frac{m}{1}$ We need, for example, a rational extension of 7 so that we can combine an integer by an arithmetic operation with a rational number; $7 \times \frac{2}{3} = ?$, but $\frac{7}{1} \times \frac{2}{3} = \frac{7 \times 2}{1 \times 3} = \frac{14}{3}$. Nevertheless, we feel free to write in short $7 \times \frac{2}{3}$ informally.

A rational number $\frac{m}{n}$ is placed on the number line by subdividing the intervals between integers into n parts and counting off m of them starting at the origin. For example, $\frac{5}{3}$ is placed on the number line by subdividing the integer intervals into 3 equal parts and counting off 5 of the subdivisions to the right from the *origin*, 0.



The square root of a rational number may not be a rational number, but rather a nonrepeating, unending decimal. For example

$$\sqrt{2} = 1.414213562373095048 \cdots.$$

The ***irrational numbers*** are the nonrepeating, unending decimal numbers.

Examples of irrational numbers

$1.01001000100001 \cdots$ $\qquad \pi = 3.141592653589793238 \cdots$

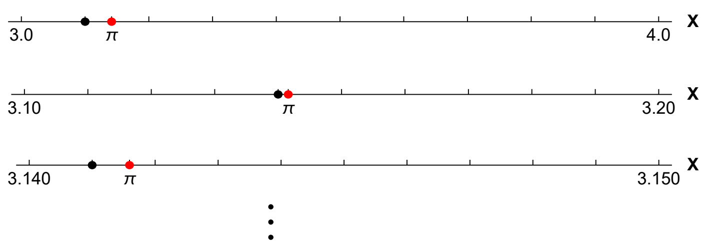
$1.23456789101112 \cdots$ $\qquad \sqrt{\pi} = 1.772453850905516027 \cdots$

$1.38159834725918 \cdots$ $\qquad \sqrt{\frac{22}{7}} = 1.77281052085583665 \cdots$

Irrational numbers are difficult to place on a number line. What we do is approach the location *exactly* by an unending sequence of increasing rational numbers (as suggested by its unending decimal form). We know how to place rational numbers on a line. So in a theoretical way, we can also place irrational numbers on a number line exactly with an unending sequence of steps.

For example,

$$3.1, 3.14, 3.141, 3.1415, \cdots = \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \frac{31415}{10000}, \cdots \to \pi.$$



The symbol $\to$ is read `approaches (exactly)'. The word *exactly* is appropriate because at each step the quality of the approximation increases by one decimal place and the unending sequence of approximations ultimately gives $3.1415 \cdots = \pi$ exactly.

The Real Extensions of Rational Numbers If you wish to combine a rational number with an irrational real number (by addition, say), you must in theory write the rational one in unending decimal form.

Every rational number can also be written an unending repeating decimal ore sometimes as a terminating decimal.

$$\frac{3}{4} = 0.75 = \frac{7}{10} + \frac{5}{10^2} = 0.75000 \cdots$$

$$\frac{7}{11} = 0.636363 \cdots = \frac{6}{10} + \frac{3}{10^2} + \frac{6}{10^3} + \cdots.$$

This is because to write $\frac{p}{q}$ in decimal form you use long division; if the remainder at any step is 0, the division stops and the result is a terminating decimal. Otherwise the remainders can only be $1, 2, \cdots, q-1$. So after at most $q-1$ steps, a remainder repeats and the result must be a repeating decimal repeating in groups of at most $q-1$ digits. For $\frac{3}{4}$, the decimal form terminates; $\frac{7}{11}$ repeats in groups of two; $\frac{2}{7}$ repeats in groups of six (verify this). Again we say the real extension of $\frac{3}{4}$ is $0.75000 \cdots$; with it you can combine $\frac{3}{4}$ arithmetically with a *real number*, any number that can be written as an unending decimal.

Example $\frac{3}{4} + \pi = 0.750000000 \cdots + 3.141592653 \cdots = 3.891592653 \cdots$

The fractional form of a repeating decimal can always be recovered.

Example

$7.4235235235\cdots$; call it x. Then

$$\begin{array}{rl} 1000x = & 7423.5235235\cdots \\ (-)\quad x = & 7.4235235\cdots \\ \hline 999x = & 7416.1 \end{array}$$

So $x = \frac{7416.1}{999} = \frac{74161}{9990}$.

Terminating decimals, other than for 0, can be written in repeating decimal form in two ways:

$\frac{3}{4} = 0.75 = 0.75000\cdots$

$\frac{3}{4} = 0.75 = 0.74999\cdots$.

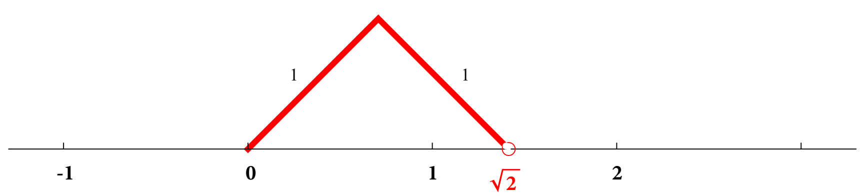
The set of *real numbers* $\mathbb{R}$ is the set of all unending decimals. That is, every real number r can be written in the form

$$r = \pm n.d_1 d_2 d_3 \cdots$$

where n is zero or a positive integer.

Natural numbers, integers and rational numbers are in the set of real numbers because their *real extension* can be written in repeating decimal form; however, one may feel free to write them in their usual natural number, integer, or rational (fractional) form.

When all the real numbers have been placed on the number line we obtain the ***real number line***. The real number line is perfect for measuring because every physical measurement, as far as we know, is a real number. We say `the real line is *geometrically complete.*' The rational line is not geometrically complete; for example, with it we would not be able to measure exactly the hypotenuse of the right triangle shown below because the rational line does not have a number at $\sqrt{2}$.



Note: at the elementary level you cannot take the square root of a negative number or the logarithm of 0 or a negative number; however within the set of *complex numbers* $\mathbb{C}$, $\sqrt{-1} = i$ and $\log_e(-1) = i\pi$, as examples. You cannot ever divide by 0 or take the log of 0.

The real numbers are also algebraically complete because every legal arithmetic operation at the elementary level gives a real number.

Are there other numbers? Yes there are. We do not need any more for real world calculations or measurements. However the inventors of calculus in the seventh century found out that the theory and calculations of calculus would be easy if very small numbers called *infinitesimals* existed. At that time infinitesimals were not known to exist. But they, without any confirmation of their existence, used them anyway and quickly discovered most of the calculus you will learn this year.

In the mid-nineteenth century mathematicians discovered the rigorous but difficult epsilon-delta calculus which gave calculus the reputation of being a very difficult subject.

About 1960, the mathematician Abraham Robinson proved that the earlier infinitesimals did exist. Some day infinitesimal calculus will be widely used again!

Exercises

Read the lesson very carefully. Make sure you understand everything. However, there is no need to memorize much.

Try all the exercises below. Do not look at the solutions except to check your answers or if you need a hint.

1. Write each integer as a rational number in two ways.
   a. 7
   b. –3

2. Write each as a terminating decimal. Use long division.
   a. $\frac{3}{4}$
   b. $\frac{7}{10}$
   c. $\frac{4}{25}$

3. Write each in an unending decimal form.
   a. $\frac{4}{9}$
   b. $\frac{4}{11}$
   c. $\frac{4}{7}$

4. Write each in fraction form.
   a. $0.999\cdots$
   b. $0.373737\cdots$
   c. $71.333141414\cdots$

5. Write a sequence of rational numbers approaching $\sqrt{5} = 2.23606\cdots$.

6. Use the sequence of #5 to plot $\sqrt{5}$ within 0.001 of its correct place on the real line.

7. The theorems, Further Properties of Equals are often, but improperly, restated as `you can do the same thing to both sides of an equation' without changing its solutions. Show that this not true for squaring. That is $a = b \iff a^2 = b^2$ is not true.

8. Determine which of the following are irrational real numbers.
   Hint: is the number likely to be a non-repeating decimal?
   a. $\sqrt{\frac{9}{4}}$
   b. $\sqrt[3]{27}$
   c. π^3
   d. 0.767667666 · · ·
   e. 7.010101 · · ·

9. Find the decimal expansions of $\frac{1}{n}$ for n from 1 to 11. Identify the group of repeating digits for each.

10. Invent three examples of irrational numbers in decimal form which are easy to memorize.

11. Invent a right triangle with hypotenuse 5. Is there one with hypotenuse 3 ?

12. What are the integer, rational and real extension of the natural number 5?

13. Use appropriate extensions to work each.

    a. $13 + \frac{1}{4}$. Do with and without decimal representations.

    b. $\pi + \frac{2}{9}$

    c. $\frac{1}{3}$ + 0.1234567891011 · · ·.

14. Which is larger: 36/45 or 37/46?

15. Which is larger: 7.532438 or 7.532418?

Solutions

1\. a. $7 = \frac{14}{2} = \frac{-42}{-6}$

3\. a. 0.444444 · · ·
   c. $\frac{4}{7}$ = 0. 571428571428 · · ·

4\. a. 1
   c. $\frac{7061981}{49000}$

5\. 2, 2.2, 2.23, 2.236, · · ·

7\. For example, the equation:

$x = 3$ has the solution set {3}. Squaring:
$x^2 = 9$ has the solution set {-3, 3}.

11\. For example, 1, 2, $\sqrt{5}$ or $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$; 1, $\sqrt{2}$, $\sqrt{3}$

13 b. $\pi + \frac{2}{9}$ = 3.14159265 · · · + 0.22222222 · · · = 3.36381487 · · ·.

APPENDIX Some Algebra Reminders

The rules of algebra are the same as the rules of arithmetic:

Associative Laws for + and ·
Commutative Laws for + and ·
Distributive Law
Existence of Identities 0 and 1
Existence of inverses -x and $\frac{1}{x}$, $x \neq 0$.

Keeping these in mind helps prevent algebra mistakes.

Order of Operations

To avoid excessive use of parentheses, obey the following conventions.

First: evaluate inside parentheses (including those implied by arguments of functions, roots, exponents and fractions)
Then: do multiplications and divisions
Then: do additions and subtractions

Bad and Good Algebra

We conclude with a list of common algebra errors along with their correct counterparts. These mistakes count ***doubly wrong*** on an exam!

False Linearity

$$\sqrt{x^2+y^2} \neq x+y$$
$$f(x+y) \neq f(x)+f(y)$$
$$\sin(x+y) \neq \sin x + \sin y$$
$$b^{x+y} \neq b^x + b^y$$

Only Correct Case (linear function)

$g(x) = kx$, where k is a constant.

$$\Longrightarrow g(x+y) = g(x) + g(y)$$

Wrong Fraction Property

$$\frac{a}{b}+\frac{a}{c} \neq \frac{a}{b+c}$$
$$\frac{a}{b}+\frac{c}{d} \neq \frac{a+c}{b+d}$$

Correct

$$\frac{a}{c}+\frac{b}{c} = \frac{a+b}{c}$$
$$\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}$$

Improper Cancellation

$$\frac{2^x+x-3}{x+2} \neq \frac{2^x-3}{2}$$

Correct: Remove a Common Factor, i.e. Cancel

$$\frac{x(x+2)}{x(x^3+3x-2)} = \frac{x+2}{x^3+3x-2},\ x \neq 0$$

Others

$$\sqrt{x^2} \neq x \text{ unless } x \geq 0$$
$$\sqrt{x^2} = \sqrt{(-x)^2} \neq -x \quad \textbf{N.B.}$$

Correct

$$\sqrt{x^2} = |x|$$

Example $\sqrt{9} = \sqrt{(-3)^2} \neq -3$

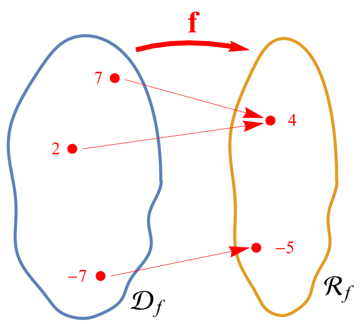
$$\sqrt{9} = \sqrt{(-3)^2} = |-3| = 3$$

0.2 What is a Function?

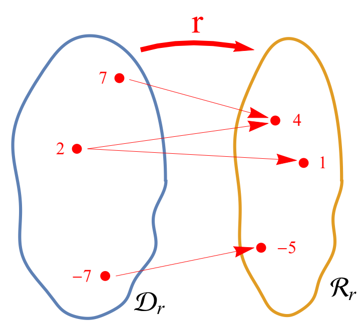
Functions are the fundamental objects we study in calculus; so we need to know exactly what a function is. Here we review the basics.

Definition A ***function*** f associates every number x in its ***domain*** set $\mathcal{D}_f$ exactly one number y = f(x) in its ***range*** set $\mathcal{R}_f$.

Examples



f **is *a* function**



r **is not *a* function**

Ways of Representing a Function

I. Functions as data or table of values

Example 1 The position x of a cart at times t is show on the table below.

T seconds	1	2	3	4	5
X meters	1	2	4	7	11

The domain is the set {1, 2, 3, 4, 5].

The range is the set (1, 2, 4, 7, 11}.

For each t there is exactly one x.

$\Longrightarrow$ The data defines a function.

Note that in most experiments, for each measurement t, there is *exactly one* result y.

That is why the 'exactly one' restriction in the definition of a function.

Example 2. The function f diagrammed above

- f = {{ -7, -5}, {2 , 4}, {7 , 4}}
  It is a function function because no different ordered pairs have the same first element.

Example 3. The non-function r diagrammed above

r = {{ -7, -5}, {2 , 4}, {2 , 1}, {7 , 4}}

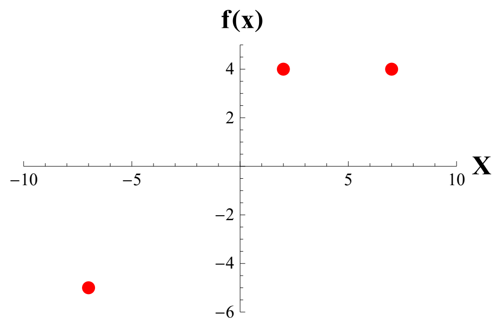
is not a function because two different ordered pairs have the same first element.

II. Functions as graphs

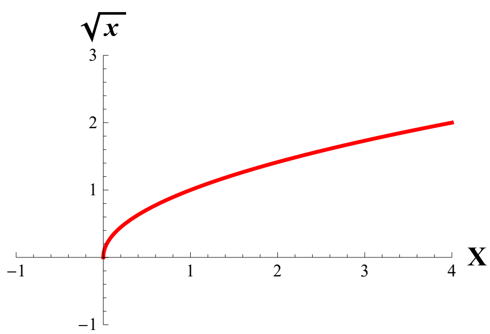
We associate an ordered pair {x, y} of a function with the point (x, y) and graph on a rectangular coordinate system. The 'only one' function requirement means it passes the '***vertical line test***.'

For many people graphs are preferred, especially those not in the physical sciences. "A graph is worth a thousand x's."

Let us look at Example 2 above in graphical form.



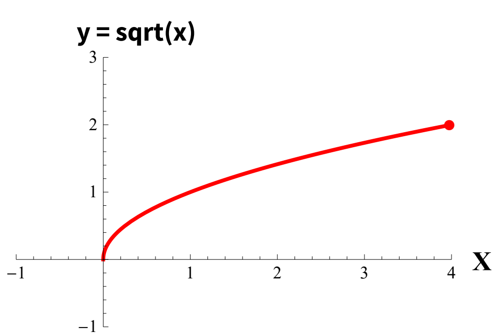
Example 4 The Square Root function



At x = 4, the absence of a large dot means the graph extends to the right. No domain was specified, so we assume it is its **natural domain**, $x \geq 0$.

III. Functions as a formula + a domain

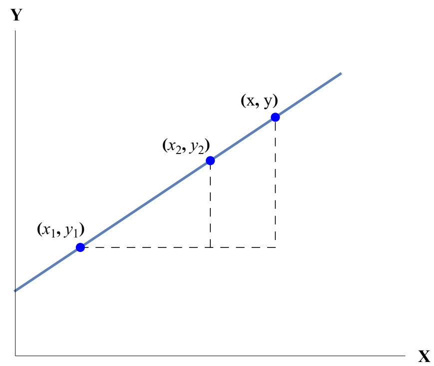
Example 5. sqrt(x) = $\sqrt{x}$, $0 \leq x \leq 4$.
This function is different than the previous function because it has a different domain.



Linear Functions

Linear functions are as important in calculus as they are in other areas of mathematics. As a review, we will do a derivation of a line through two points.

Let the two points be (x_1, y_1) and (x_2, y_2). Let (x, y) be any point on the line.



Then by similar triangles

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

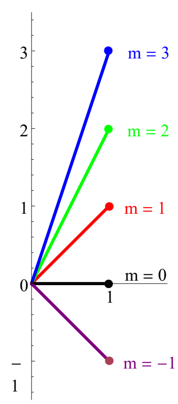
or

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Two Point Form of a line

Definition ***Slope*** $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

To draw a line of slope m, start at a point on the line. Go 1 unit in the x-direction and m units in the y-direction. Mark the new point. Then draw the line through the two points.



Substituting m into the two-point form, we get perhaps the most important for calculus

$$y - y_1 = m(x - x_1)$$

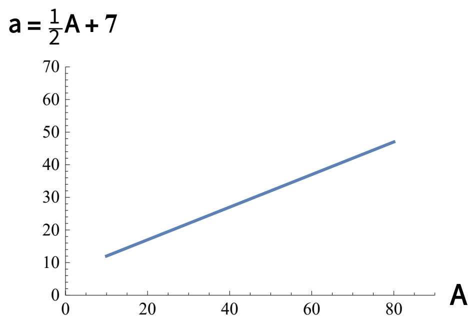
Point-Slope Form

Substituting $(0, b)$ for (x_1, y_1), we get

$$y = m x + b$$

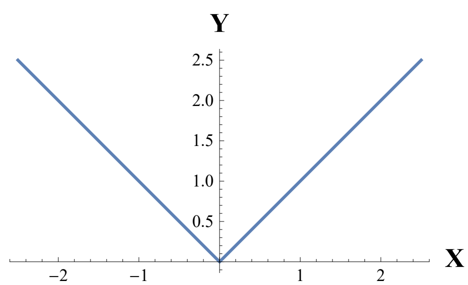
Slope-Intercept Form

Example 6 Minimal Decency Curve It is minimally OK for a person of age A to date someone whose age is given by $a = \frac{1}{2}A + 7$.



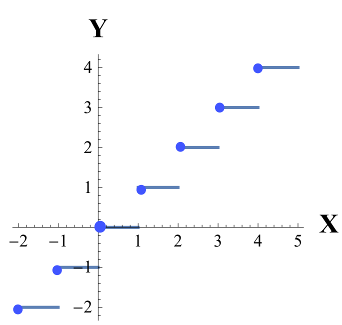
Wikipedia The “never date anyone under half your age plus seven” rule is a rule of thumb sometimes used to prejudge whether an age difference is socially acceptable. Although the origin of the rule is unclear, it is sometimes considered to have French origin.

Example 7 Absolute Value Function $y = |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$.



Absolute values are used when only the size or magnitude of a quantity matters.

Example 8 The Floor Function y = Floor(x). The Floor function rounds down to the nearest integer.



Jumps like these occur, in theory, whenever data is obtained from a digital readout.

Exercises

1. Find the equation of each line.
   a. The line through the points (1, 2) and (-2, 3).
   b. The line through (2, 3) with slope -2.
   c. The line with y-intercept 3 and slope -1.

2. Show that $\frac{x}{a} + \frac{y}{b} = 1$, the *intercept-intercept form* of a line, has the x-intercept a and y-intercept b.

3. Show the details of the simplifications of the two point form of the line to the other forms

4 Graph the absolute value function, $y = |x|$. Make a table of values.

5. Find the relationship for the temperature F in °F in terms of the temperature C in °C.
When is $F = C$?

6. Suppose hamburger casts 1 cent for each full gram purchased (digital readout).
   a. How much does it cost if you get 0.5 grams for your pet roach?
   b. Draw an accurate graph (price vs grams) for purchasing up to 5 grams.

7. For the basic decency curve, when is $A = a$?

8. What do the results of a lab experiment using a digital readout have to do in common with the Floor function?

9. A box with a square base x cm by x cm and an open top has a volume of 100 cm^2. Find the function which gives its surface area.

10. Derive the mid-point formula for the points (x_1, y_1) and (x_2, y_2).

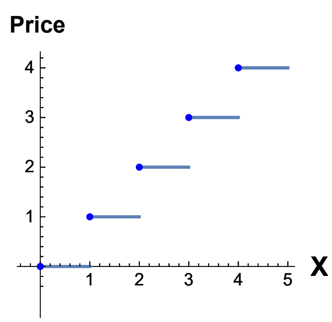
Solutions

5. Hint

°C	°F
0	32
100	212

6. a. 0 cents

b.



7. $a = \frac{1}{2}A + 7$, $A = a$

$A = \frac{1}{2}A + 7$

$\frac{1}{2}A = 7$

A = 14 years. 8^{th} graders should be careful.

9. $A = x^2 + \frac{100}{x}$

0.3 Memory Functions and Operations on them

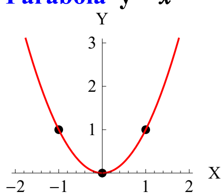
You have encountered many examples of functions in high school and your leisure reading. It is time to review some of them and reincorporate them into your active collection of recognition functions. Look at each graph/equation pair and identify its interesting features.

These graphs you should be able to graph quickly and fairly accurately with a short table of values as shown by points on the graphs:

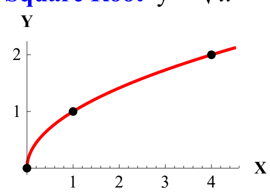
Two data points for each line segment. Three data points for each 'hump' or curved segment.

You are expected to be able to graph these quickly and fairly accurately on exams.

Parabola $y = x^2$



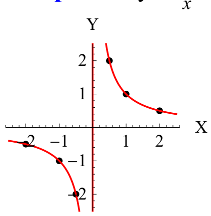
Square Root $y = \sqrt{x}$



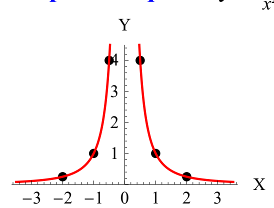
Think 'curved segment':

x	y
0	0
1	1
4	2

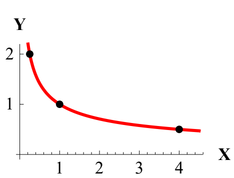
Reciprocal $y = \frac{1}{x}$



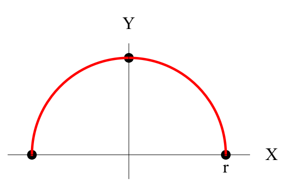
Reciprocal Square $y = \frac{1}{x^2}$



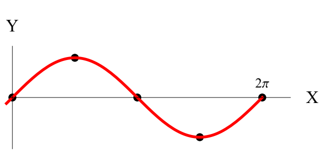
Reciprocal Square Root $y = \frac{1}{\sqrt{x}}$



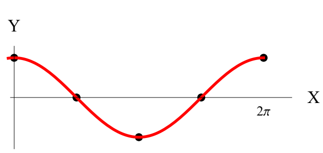
Semicircle $y = \sqrt{r^2 - x^2}$



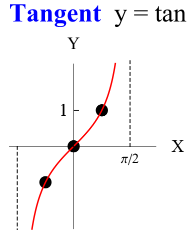
Sine $y = \sin x$



Cosine $y = \cos x$



Tangent $y = \tan x$



Memory Work

Be able to sketch the above functions quickly. These functions are ones you will be required to sketch rapidly and fairly accurately in exams.

Transforming and Combining Functions

Chemistry is easy. There are only about 100 elements and a handful of ways to combine them. Not quite true! But this idea is even truer for basic functions study. There are nine functions on the above memory list. Today we will look at a few ways of transforming and combining those functions so that we greatly expand the number of functions we can readily graph. These transformation methods are very important.

Transforming functions

Translation (or Shifting) Principle

$F(x, y) = 0$ shifted

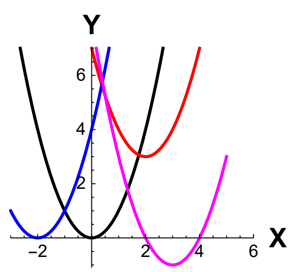
$\begin{cases} h \text{ units horizontally} \\ k \text{ units vertically} \end{cases}$

becomes

$F(x - h, y - k) = 0$

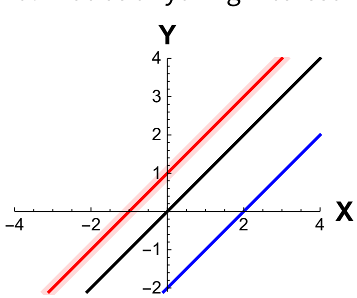
Example Start with a memory friend, the parabola $y = x^2$ (graphed black).

a. Shift 2 to the right, 3 up: $y - 3 = (x - 2)^2$. (graph red)
b. Shift 2 to the left: $y = (x + 2)^2$. (graph blue)
c. Shift 3 to the right, 1 down. (graph magenta)



Example An easy, but important one: y = x (graphed black).

a. Shift 3 to the right. (graph blue)
b. Shift 1 up. (pink)
c. Shift 1 to the left. (red)
d. Notice anything interesting?



So if you know the line y = x, you know the equation of this line shifted in any direction.

Stretching (or Scaling) Principle

$F(x, y) = 0$
stretched

$\begin{cases} A \text{ times horizontally} \\ B \text{ times vertically} \end{cases}$

becomes

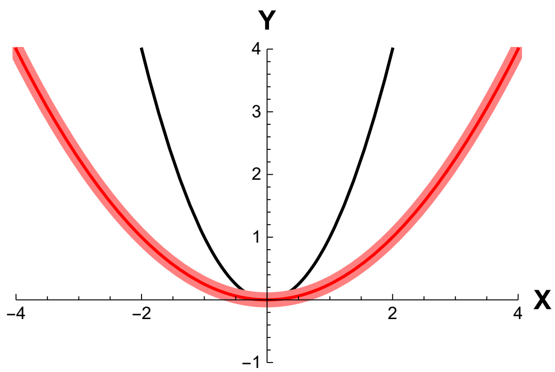
$F\left(\frac{x}{A}, \frac{y}{B}\right) = 0$

Example Start again with the parabola $y = x^2$ (graphed black).

a. Make me twice as fat: $y = (\frac{x}{2})^2$. (graph pink)

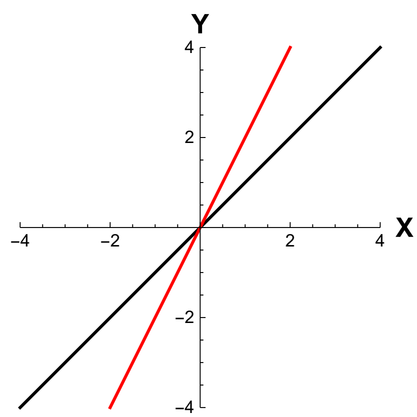
b. Make me one-quarter as tall: $\frac{y}{1/4} = x^2$ (graph red)

c. Notice anything interesting?



Example Your second favorite curve, the line: $y = x$ (graphed black).

a. Stretch by 2 vertically. $\frac{y}{2} = x$ (graphed red)



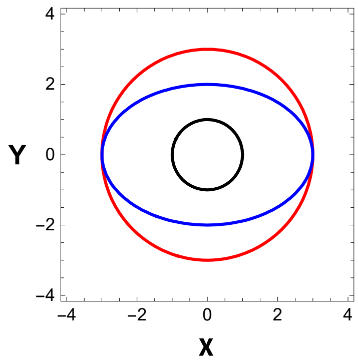
So now you only need to know only one line $y = x$! All others of any slope you can get by a shift and/or a stretch!

Example This time your favorite non-function, the unit circle about the origin.

$x^2 + y^2 = 1$ (graphed black).

a. Make me thrice as big: $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$ (graph red)

b. Stretch me by 3 times as wide and twice as high : $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ (graph blue)

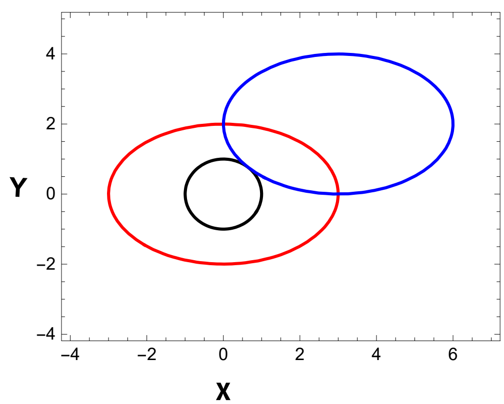


Example Yes, you can combine a stretch with a shift (that order is best).

a. Start with $x^2 + y^2 = 1$ (graph black).

b. Stretch me by 3 times as wide and twice as high : $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ (red)

c. Shift 3 to the right and 2 up: $\left(\frac{x-3}{3}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 1$ (blue)

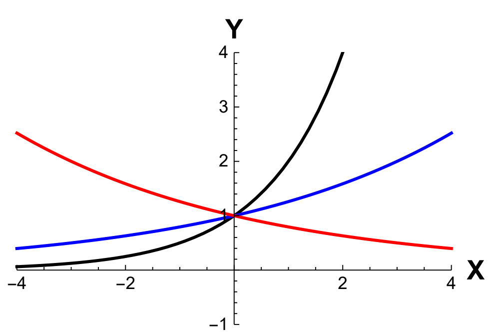


Example A bonus. If you stretch by a negative number, that corresponds to a stretch plus a flip. (Think about why this is true)

a. Start with the exponential function $y = 2^x$. (black)

b. Stretch by 3 horizontally. $y = 2^{\frac{x}{3}}$ (blue)

c. Flip across the y axis. $y = 2^{-\frac{x}{3}}$ (red)



Combining Functions

Another way of gaining expertise at new functions is to combine two or more known functions by basic algebraic operations

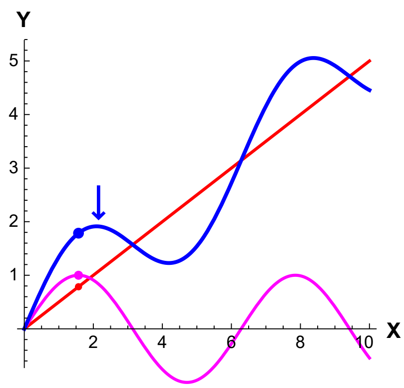
If f(x) and g(x) are functions you know, then so are by

Addition	f(x) + g(x)
Subtraction	f(x) - g(x)
Multiplication	f(x) · g(x)
Division	f(x) ÷ g(x)
Composition	f(g(x)).

We know a function well if you can compute with it or, perhaps even better, graph it by hand quickly. Some tools are graphical addition, subtraction, multiplication and division. Even graphical composition is possible.

Example Graphical addition.
Graph $y = x/2 + \sin x$

First graph $x/2$ and $\sin x$ separately.
Then add corresponding y-values.



Quite frankly, these are often hard to do by hand. I would do it here by noting where $\sin x$ is 0 or has a high or low point and plotting those points and connecting them with a reasonable curve. It is easy to make mistakes. You might expect a high point on the sum curve to be at the high point of the sin curve. Not true!

Buy a graphing calculator! However, knowing about how these combinations work is often useful in analyzing graphs.

Exercises

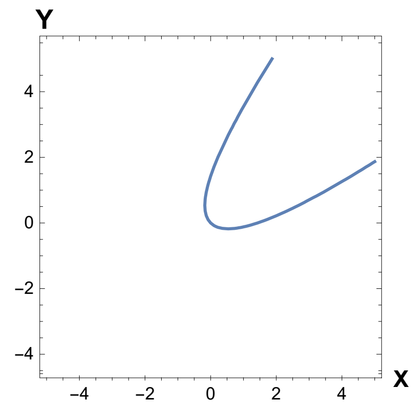
1. $y = \sqrt{1 - x^2}$, upper unit semicircle. Find the equation for each transform. Graph each.
   a. shift 1 to the right.
   b. shift 3 to the left, 2 up.
   c. stretch by 2 horizontally.
   d. stretch by 2 horizontally, -3 vertically.

2. $y = \begin{cases} 0, & x \le 0 \\ x, & 0 < x \le 1 \\ 2 - x, & 1 < x \le 2 \\ 0, & x > 4 \end{cases}$ Find the equation for each transform. Graph each.
   a. shift 1 to the right.
   b. shift 3 to the left, 2 up.
   c. stretch by 2 horizontally.
   d. stretch by 2 horizontally, -3 vertically.

3. $x^2 + y^2 = 4$. Find the equation for each transform. Graph each.
   a. shift 2 to the right and 2 up.
   b. compress the circle of part by a factor of 2 vertically,

4. $f(x) = 2x + 3$. $g(x) = \sqrt{1 - x^2}$. Find
   a. $f(g(x))$
   b. $g(f(x))$
   c. $f(f(x))$
   d. $f(g(f(x)))$

5. Use graphical subtraction to graph $y = \sin x - \frac{x}{3}$.



6. The transformation

$$x \longrightarrow 0.7\,X - 0.7\,Y$$
$$y \longrightarrow 0.7\,X + 0.7\,Y$$

rotates the parabola $y = x^2$. For further enlightenment on this, take a Linear Algebra course.

7. For each of the nine graphs on your memory list, do a shift or a magnification and then graph the result. Be creative.

8. Graph $y = \frac{1}{2}x - \sin x$ by graphical subtraction.

9. Graph $(y - x)(x^2 + y^2 - 1) = 0$.

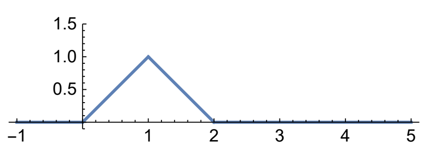
10 a. Prove the Translation Principle.
   b. Prove the Stretching Principle.

10. Graph:
   a. $y = x(x - 1)$
   b. $y = |x(x - 1)|$
   c. $y = \sqrt{x(x - 1)}$

11. The domain of $y = f(x)$ is $a < x \leqslant b$ and its range is $c < y \leqslant d$.
   a. What is the domain and range of $y - k = f(x - h)$?
   b. What is the domain and range of $y/B = f(x/A)$?

Solutions

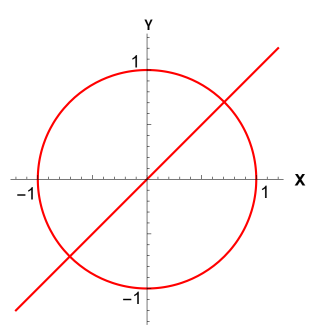
#2.



#4. a. $f(g(x)) = f(\sqrt{1 - x^2}\,) = 2\sqrt{1 - x^2} + 3$
   b. $g(f(x)) = g(2x+3) = \sqrt{1 - (2x + 3)^2}$
   c. $f(f(x)) = f(2x+3) = 2(2x+3)+3$

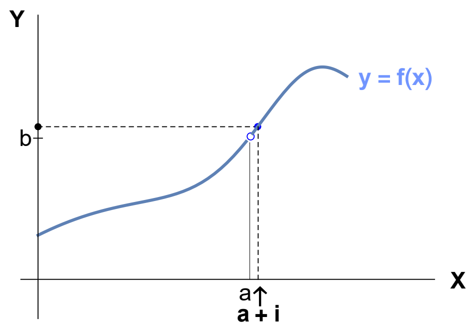
#8. Hint: graph $y = \frac{1}{2}x$ and $y = \sin x$ separately and subtract suitable y-values.

#9. Hint: $AB = 0$ implies $A = 0$ or $B = 0$.



0.4 Discovering Infinitesimals. Counting to Infinity.

You encounter an unusual mathematical problem when starting calculus. It is will be often necessary there to evaluate a function which is undefined precisely at the point of interest. Look below for such a function at x = a.



You will need to know the value of f at x = a, but f(a) does not exist. What will suffice is the value b, if it exists, as suggested by f(x) when x is *very close* to a; unfortunately, 'very close' is not a precise or easily quantifiable idea in terms of real numbers. The discoverers of calculus, particularly the seventeenth century co-discoverer of calculus, Gottfried Wilhelm Leibniz, took 'very close' to mean any nonzero ***infinitesimal*** distance ***i*** from the point a. Infinitesimals were thought to be some strange kind of number smaller in size than any positive real number. To find b he calculated f(*a* + ***i***) for every nonzero infinitesimal ***i*** and after doing some algebra with them, set ***i*** = 0. He did not know what an infinitesimal was or even if such a number existed; furthermore, how could ***i*** be non-zero and then take it to be zero?

Nevertheless, despite the lack of clarity about what infinitesimals were, mathematicians then were skilled at doing the relatively easy, direct calculations desired of them and in short order discovered most of the calculus formulas, theorems and techniques you are likely to need for elementary applications.

But still, mathematicians were quite apprehensive about their lack of understanding of infinitesimals. Imprecise ideas like those infinitesimals have no place in subject like mathematics; one cannot trust the outcome of calculations based on a vague, imprecise foundation. For applications, trust is absolutely necessary because much of modern science and technology depends on the methods of calculus.

About two centuries after the discovery of infinitesimal based calculus, the mathematician Weierstrass and others discovered the so-called ***ϵ-δ limit method*** of doing calculus. While it was rigorous and did not use infinitesimals, its adoption made the theory of calculus very difficult for beginners because it did not provide a direct calculation method for determining the number b; one had to first guess b and then verify that it was correct by solving often difficult inequalities involving absolute values. Proofs of some important calculus formulas and theorems were too difficult to put even in the appendix of textbooks; derivations of some important application techniques were needlessly complicated.

In 1960, the mathematician Abraham Robinson showed infinitesimals had a rigorous basis. However, in demonstrating this, he had to use very advanced abstract mathematics unsuitable for beginning calculus students.

Since infinitesimals make the theory, calculations and applications of calculus relatively easy, we wish to base our understanding of calculus in terms of infinitesimals and related numbers in a rigorous but intuitive way. We will begin with a search for infinitesimals. Then, after we find infinitesimals, we will look for an infinite positive integer. Finally, with this infinite integer we will be able to write infinitesimals and other related new numbers in decimal form. The decimal form of the new numbers makes them feel less abstract and immediately allows us to identify their algebraic properties and how to use them in the analysis of functions.

A preliminary concept - the cardinal number of a set

The natural numbers were defined in terms of an intuitive idea of the sizes of sets. We will extend that idea to non-finite sets where we use the term ***cardinal number*** or ***cardinality*** for the number of elements in such sets.

Let us begin by looking at the set whose elements form an unending sequence

$\{a_1, a_2, a_3, \cdots a_n, \cdots\}$.

The number of elements in this set is called $\aleph_o$, ***Aleph-zero***. It is the smallest infinite cardinal number. An important principal when working with cardinal numbers is:

If the elements of two sets can be put into a 1-1 correspondence, then the sets have the same cardinal number.

For example, the sets $\{1, 2, 3, \cdots n, \cdots\}$ and $\{2, 4, 6, \cdots 2n, \cdots\}$ both have the same cardinality $\aleph_o$ because their elements, can be put into a 1-1 correspondence

$1 \leftrightarrow 2,\ 2 \leftrightarrow 4,\ 3 \leftrightarrow 6,\ \cdots\ n \leftrightarrow 2n,\ \cdots$.

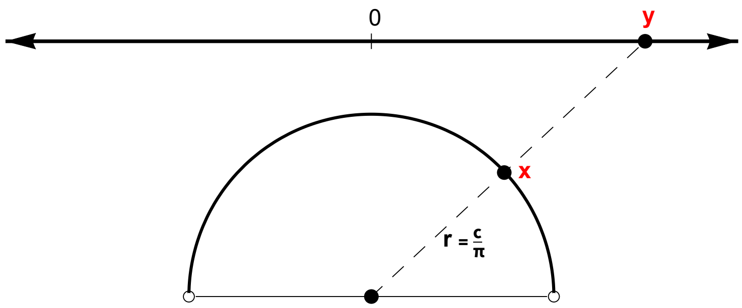
This may seem counterintuitive because the first set appears to have more elements than the second, but it is according to the principle nevertheless correct and is widely used in advanced mathematics.

As a side comment, mathematics problems are often are often categorized as being either easy or hard. An ***easy problem*** is one which can be done in a finite number of steps. Solving a quadratic equation is an easy problem because its solution can be found using the quadratic formula, which requires only a few steps including simplification.

A ***hard problem*** is one which requires an unending sequence of better and better approximations which approach the exact solution. Solving a fifth degree polynomial equation is often a hard problem. For example, $x^5 - x + 1 = 0$ is a hard problem. A more elementary example is finding the square root of 2 in decimal form. One way of doing this is by trial and error and with the aid of a calculator, finding the largest n significant digit decimal number whose square is less than 2 for n = 1, 2, 3, $\cdots$. When you do this, you get 1, 1.4, 1.41, 1.414 $\cdots$ which after $\aleph_o$ steps gives you $\sqrt{2} = 1.414213562\cdots$ exactly.

The set of all real numbers or equivalently the set of unending decimal numbers, $-\infty < x < +\infty$, does not have cardinality $\aleph_o$; it has a larger cardinality called $\mathfrak{c}$ (for ***continuum***). Real numbers often result from unending sequences of rational numbers. The real numbers also are required for space or time variables and many other measurable physical quantities.

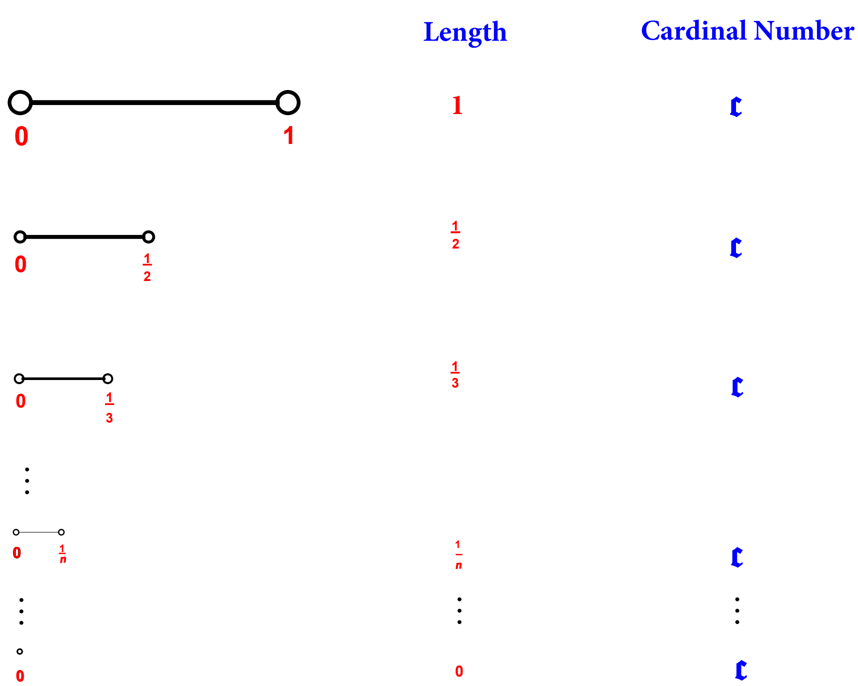
The figure below shows there is a 1-1 correspondence between each point x on a semicircle of length c and a point y on the real number line. Thus the open interval $0 < x < c$ has the same cardinality $\mathfrak{c}$ as the entire line of real numbers!



Infinitesimals Exist

A ***positive infinitesimal*** ***i*** is a number which satisfies $\mathbf{0 < \mathit{i} < \frac{1}{n}}$ for every natural number n.

Let us hunt for infinitesimals by considering the sequence of intervals below.



As *n* increases through the natural numbers $\mathbb{N}$:

The length of the open intervals decreases to 0.

But the cardinality of each interval remains $\mathfrak{c}$.

The end result is an open interval of length 0.

It contains no real numbers.

⟹ The numbers remaining must be infinitesimals!

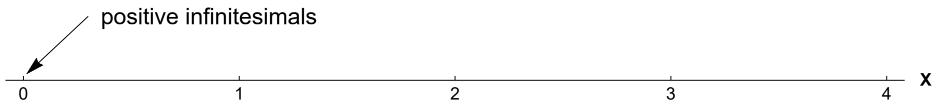
NOTE I find this argument entirely convincing but could not find any support for it in the literature.

Again: **The end result of going down the sequence is an open interval of length 0 and cardinality ¢, whose elements are all smaller than $\frac{1}{n}$ for every natural number n, and which therefore must be infinitesimals*!

This observation indicates there must be a continuum of infinitesimals just to the right of the origin and to the left of every positive real number as shown below. This prompts the following axiom.

Axiom Infinitesimals exist.

Note **You don't actually have the option of not accepting the existence of infinitesimals. Each infinitesimal exists and has a unique place on the number line. Shortly you will learn how to place them there.**



We could proceed directly with these abstract infinitesimals to construct a new so-called *hyperreal number system* for doing calculus. However, it will be useful to write infinitesimals and other hyperreal numbers in a decimal form in order to get an intuitive concrete feeling for these new numbers and to help discover their algebraic properties and how they are used in the analysis of functions; we will need to find a positive *infinite integer* in order to do this. So we will start by looking for such a very large number (since the reciprocal of very small positive real numbers are very large positive real numbers, we should suspect the reciprocals of positive infinitesimals to be infinitely large positive numbers and that some of these might be infinitely large positive integers

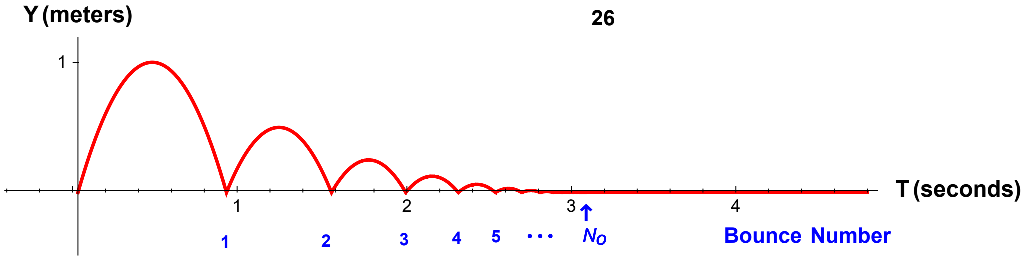
A positive infinite integer exists! **A thought experiment** The idea of experiencing an infinite number of events, particularly in a finite time period, might be hard for you to conceive. Imagine that you throw a ball up to a height of 1 meter and that after each time it hits the ground, it bounces up to exactly half its previous bounce height. Clearly the ball does an unending (as opposed perhaps to an actual infinite) number of bounces; every bounce is followed by another bounce half as high. You might think that the ball bounces forever, in theory, and never comes to a complete stop. Surprisingly, the bouncing lasts only for about 3.08 seconds! (You can show this if you know a bit of physics, $y = \frac{1}{2} g t^2$, and the geometric series

$1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}$ if $|r| < 1$. See Exercise 2.)

If you had perfect real hearing or vision, you should be able to hear or see the unending sequence of real number height bounces.

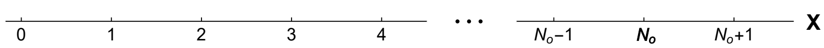
Now that we know that infinitesimals exist, the ball after it stops making real bounces continues, of course, with bounces of infinitesimal height for a further infinitesimal period of time (how would it know not to do so!). If you had hyper-hearing, you would hear an *actual infinite number* of bounces. (This is a thought experiment for a classical Newtonian ball, the kind you normally think about; so for this thought experiment we will ignore the physical fact that quantum mechanics for this bounded system forbids arbitrarily small bounces.) The next figure shows the height of our bouncing ball as a function of time.

Note again **I'm not certain that all mathematicians would consider our 'demonstration' of the existence of an infinite integers rigorous. My students and I found it convincing. I saw no mention of thisin the literature. If it incorrect, we would simply postulate their existence on the basis of Abraham Robinson's work, but not have much intuition about them.**



Next, let us construct a number line with an actual infinite number of integers marked off on it. Here is how you can do it; using our bouncing ball as a metronome will make counting to infinity seem intuitive and easy. Start at a point marked 0. Throw the ball up to a height of 1 meter. Every time you hear a bounce, mark off other integers at equal spacings and label these appropriately as 1, 2, 3, · · ·. After you have recorded the real height bounces, you will have the familiar unending positive real integer line. Continue marking off until you hear or see (with your imagined hyper-hearing or hyper-vision abilities) infinitesimal height bounces; stop after one of these bounces and record it as the ***infinite integer*** $\boldsymbol{N_o}$. (For later convenience, we will want N_o to be an *even* infinite integer; you can assure this by counting off the bounces in groups of two.

Then, continuing this process of marking off the bounces, you will get a positive integer line as shown below which includes the positive infinite integers. So you have found an infinite integer *and* constructed the infinitely long positive integer number line in a little more than 3.08 seconds! (This required you to travel at hyper-relativistic speeds; again, this is a thought experiment.)



The number N_o *is at a* ***definite point*** *on the line corresponding to a definite number of bounces. The integer line you just constructed records an unbroken sequence of whole numbers from 0 to* N_o *and beyond. This line at infinite whole numbers, other than the number names labeling it, looks exactly like the line at finite whole numbers.*

Infinitesimals in decimal form

Now that we have an infinite integer N_o, we can write an infinitesimal in decimal form! Consider the number

$$i_o \equiv 10^{-N_0} = \mathbf{0.000 \cdots 001,000 \cdots}.$$

For convenience, we use a comma in decimal numbers to mark off groups of N_o decimal places from the decimal point (contrast this with the ordinary comma usage used to mark off groups of 3 decimal places). $\boldsymbol{i_o}$ is an infinitesimal because it has zeros at all finite decimal places (hence it is smaller than any positive real number which would have a nonzero digit at some finite place); because of the 1 at the infinite decimal place N_o, it is also nonzero and positive. We can think of i_o as our special, basic infinitesimal. You can place it exactly on the number line by subdividing the interval $0 \leqslant x \leqslant 1$ into 10^{N_o} equal parts and then counting off one sub-interval.

Alternative Definitions $\boldsymbol{i}$ is an ***infinitesimal*** means

$$|\boldsymbol{i}| < \frac{1}{n} \text{ for every natural number n}$$

or equivalently

$$|\boldsymbol{i}| < r \text{ for every positive real number r}$$

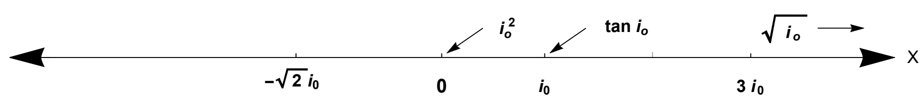
or equivalently

$\boldsymbol{i}$ is any number with zeros at all finite decimal places.

$0 = 0.000 \cdots 000, 000 \cdots$ is an infinitesimal and is the only infinitesimal that is a real number. There are many infinitesimals, both positive and negative, in addition to 0 and i_o.

Examples Infinitesimal numbers in decimal form are now easy to write down.

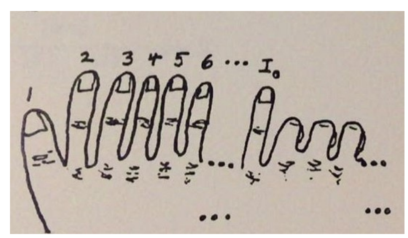
$$0 = 0.000 \cdots 000, 000 \cdots$$
$$i_o = 0.000 \cdots 001, 000 \cdots$$
$$3i_o = 0.000 \cdots 003, 000 \cdots$$
$$-\tfrac{1}{3} i_o = -0.000 \cdots 000, 333 \cdots$$
$$\sqrt{2}\, i_o = 0.000 \cdots 001, 414 \cdots$$
$$i_o^2 = 0.000 \cdots 000, 000 \cdots 001, 000 \cdots$$ The first 1 is in the $2N_0{}^{th}$ place.
$$\sqrt{i_o} = 0.000 \cdots 001 \cdots 000, 000 \cdots$$ The 1 is in the $(N_0/2)^{th}$ place; since N_0 is even.



Observe on the infinitely magnified real line that i_o^2 is much smaller than i_o and that $\sqrt{i_o}$ is much larger than i_o; these two numbers and others such as $\tan i_o$ are infinitesimals which are not real number multiples of i_o.

Note also that the term *real*, as in *real number*, refers to the set of unending decimals we introduced in section 0.1. However, the nonzero infinitesimals are also real in the sense that there is room for them on the number line. They are also not unreal in the sense that the imaginary or complex numbers of the form $a + b\,i$ ($i^2 = -1, b \neq 0$) are. They are unreal mainly because real world measurements only require real number precision.

The task in the next section is to learn how the infinitesimal numbers can be combined with the real numbers to yield the set of so-called hyperreal numbers. Then we will be ready to do calculus. Before that you will want to understand infinity.



If you had enough fingers ...

Getting Comfortable with Infinity!

In many applications, you are interested in what happens for large (infinite) values of space or time. First, how can you get there in order to get a good look at infinite places? Perhaps once when you were young you decided to run away from home. After several days of walking you realized you barely got out of the city. Perhaps if you were precocious, you wanted to get away from it all by walking to infinite places. But you eventually realized you were making almost no progress.

If you were precocious in an Einsteinian way, you may have realized the problem was with the old ticktock watch you used as a metronome to pace yourself. Then you realized that if you used our bouncing ball as a metronome, you could get to see anywhere in your infinite places in about 3.08 seconds!

To your surprise, space there looked just like back home. But the street signs were very long. A special moment was passing **1,000 · · · 000 Street**, known to the locals there at infinity as **'I_o Street'.**

With the same metronome many hard problems become easy. You can calculate and write down ***all*** the real or hyperreal digits of the square root of 2 or even π. Even easier is placing i_o exactly on the number line by subdividing the interval from 0 to 1 into I_o equal subdivisions and then counting off 1.

You now have some superpowers you may not have expected as a bonus for taking calculus!

Exercises Numbers 2, 5, 8, and 12 which are optional.

1. Which of the following are infinitesimals?

   a. 0 b. $\frac{\sqrt{3}\, i_o}{2}$ c. $-\pi i_o$

   d. $\frac{1}{10000\, i_o}$ e. 10^{-1000} f. $1000\, i_o$

   g. $\frac{i_o}{i_o^2}$ h. $\frac{i_o}{\sqrt{i_o}}$ i. i_o^{1000}

   j. 0.00000000015 k. $0.0033 \cdots 333{,}000 \cdots$ l. $0.0000 \cdots 000{,}000 \cdots 000{,}0700 \cdots$

2. a. Show that our bouncing ball stops bouncing after about 3.08 seconds. Use the formula $y = \frac{1}{2} g t^2$ from physics and the geometric series, $1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}, |r| < 1$. $g \doteq 9.80 \frac{\text{meter}}{\text{second}^2}$.

   b. Argue that the height of the N^{th} bounce, $y_N = \left(\frac{1}{2}\right)^{N-1}$, is an infinitesimal for N an infinite integer.

   c. Show that after the ball stops making real height bounces, the ball continues bouncing making infinitesimal height bounces for only a non-zero infinitesimal period of time.

3. Write each in decimal form. Note why each is an infinitesimal.

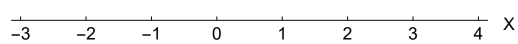
   a. $\frac{1}{2} i_o$ b. $5 i_o$ c. $\frac{1}{2} i_o + 5 i_o$ d. πi_o

   e. $2\pi i_o$ f. $(\pi + 1)\, i_o$ g. $i_o + 2 i_o^2$ h. $i_o + 2 i_o^2 + 3 i_o^3$

5. There is a website www.lightandmatter.com/calc/inf which can work problems involving i_o. In it, take d $= i_o = 0.000 \cdots 001, 000 \cdots$. See which of the numbers in Exercise 1 can be put in decimal form using this calculator; check which answers have zeros at all finite places. Also evaluate $\sin i_o$ and $\tan i_o$.

6. Explain why the set of natural number multiples of i_o, $\{i_o, 2i_o, 3i_o, \cdots\}$, is a set of infinitesimals.

7. On the line below show where the negative infinitesimals are.



8. Suppose we had chosen N_o one less and hence an infinite odd integer. Then write $\sqrt{i_o}$ in decimal form.

9. Write i_o in fraction form.

10. a. In the phrase used by the former TV broadcaster Dan Rather, 'a nit on the nut of a gnat', is the nit an infinitesimal?

    b. The shortest possible physical length is Planck's Length, 1.616×10^{-35} meter. Is this an infinitesimal

11. **Experiment** Listen to a hard ball dropped on a rigid surface. For a more dramatic and long lasting similar effect, spin a thick vertically held porcelain saucer on a hard surface to experience a similar phenomenon as it wobbles upside-down with an increasing frequency to a stop (Do this and you will appreciate why cafeterias often only give you paper plates!). Try this or view a YouTube video on ***Euler's disk***.

12. Explain why the three definitions given for an infinitesimal are equivalent.

13. **Things you can now do in 3.08 seconds** Think about these.

    a. A (seeming) paradox of Zeno says you can never go from point A to point B because you first have to go half way to B, then half of the remaining way, and so on, never getting to B. Explain why this is not actually a paradox.

    b. Describe the thought experiment for isolating the set of positive infinitesimals quickly as suggested by the idea of the demonstration of the existence of infinitesimals.

14. Suppose you do not like or believe in infinitesimals. Planck's Length, 1.6 x10^-35 m, is the shortest possible length and Planck's Time is 5.39 ×10^−44 s. Could you get away with taking an infinitesimal to be any real number less than 1×10^-1000, say, for all practical purposes?

Solutions

1. a, b, c, f, h, i, l

2. a. Time for one half-bounce:

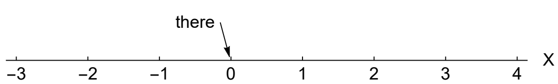
$$s = \frac{1}{2} g t^2 \Longleftrightarrow t = \sqrt{\frac{2}{g} s}\ ;\ s = \frac{1}{2^{n-1}}$$

Total bounce time:

$$\begin{aligned} T &= 2\sqrt{\frac{2}{g}1} + 2\sqrt{\frac{2}{g}\frac{1}{2}} + 2\sqrt{\frac{2}{g}\left(\frac{1}{2}\right)^2} + \cdots \\ &= 2\sqrt{\frac{2}{g}}\left(1 + \sqrt{\frac{1}{2}} + \left(\sqrt{\frac{1}{2}}\right)^2 + \cdots\right) && \text{A geometric series} \\ &= 2\sqrt{\frac{2}{g}}\ \frac{1}{1-\sqrt{\frac{1}{2}}} && \text{Sum of the geometric series} \\ &\doteq 3.0847 \text{ seconds} && g \doteq 9.80\ \frac{\textbf{meters}}{\textbf{second}^2} \end{aligned}$$

3. a. $0.000\cdots000{,}5000\cdots$ b. $0.000\cdots005{,}000\cdots$ c. $0.000\cdots005{,}500\cdots$
d. $0.000\cdots003{,}141\ \cdots$ e. $0.000\cdots006{,}281\cdots$ f. $0.000\cdots004{,}141\cdots$
g. $0.000\cdots001{,}000\cdots002{,}000\ \cdots$ h. $0.000\cdots001{,}000\cdots002{,}000\cdots003{,}000\cdots$
Each number in this exercise has zeros at all finite places and so is an infinitesimal.

4. a. F, b. F, c. T, d. F

7.


8. $\sqrt{i_o} = \sqrt{10^{-N_o}} = 10^{-\frac{N_o}{2}} = 10^{\frac{-(N_o+1)+1}{2}} = \sqrt{10}\,10^{-(N_o+1)/2} = 0.000\cdots003162\cdots$ where the 3 is in the $((N_0+1)/2)^{\text{th}}$ place.

10. a. No. b. No.

13. b. Start with the open interval $0 < x < 1$. Use our bouncing ball as a metronome for $n = 1, 2, 3, \cdots$, remove the real numbers $> \frac{1}{n}$. The result, taking only about 3.08 seconds, leaves only infinitesimals!

Review Our special hyperreal numbers:

N_o, our bouncing ball infinite integer

$I_o = 10^{N_o}$, our special infinite number for decimal hyperreals
$i_o = 10^{-N_o}$, our special infinitesimal number

References

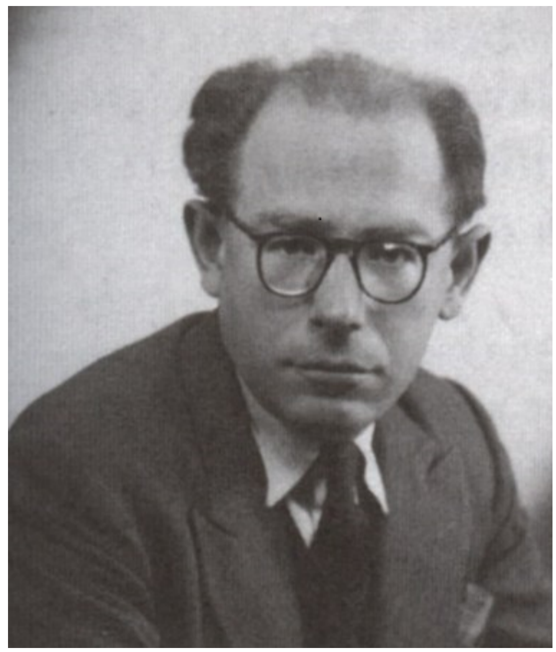
Roman Kossak, *What Are Infinitesimals and Why They Cannot Be Seen*, The American Mathematical Monthly, 1972.
James Henle and Eugene Kleinberg, *Infinitesimal Calculus*, MIT, 1980. Discusses in detail the decimal representation of hyperreal numbers. Understandable at a more advanced undergraduate level.
Jerome Keisler, *Elementary Calculus: An Approach Using Infinitesimals*, 1986, Prindle, Weber & Schmidt, Creative Commons License. This basic hyperreal calculus textbook is available free on line. It might have been more successful had been printed with color and on whiter paper!

Fathers of Infinitesimal Calculus

Gottfried Wilhelm Leibniz (1646-1716)

*Independently invented calculus using the infinitesimal based symbols we still use today. In fact, almost all the calculus you will learn was discovered using his infinitesimal based methods.

*Isaac Newton also invented calculus to solve physics problems. His calculus notation was very difficult for others to understand.



Abraham Robinson (1918-1974)

*In 1960 he showed that infinitesimals exist and they could be included in an extended real number system in a mathematically satisfactory way.

*Infinitesimal calculus now is often called non-standard analysis. A somewhat elementary presentation of non-standard calculus is given in James Henle and Eugene Kleinberg, *Infinitesimal Calculus*, MIT, 1980.

0.5 A Hyperreal Number System

For the theory of calculus it will be convenient to have a much finer and a much longer line than the real number line. To do this, we will need the infinitesimals we discovered in the last section.

We will use certain arithmetic combinations of our basic infinitesimal $i_o = 0.000\cdots001{,}000\cdots$ and the real numbers to construct a new set of numbers called the ***hyperreal numbers*** (`hyper' in this context means *more than*) which includes all these combinations; we will use these combinations to do calculus computations. Since we have the decimal form of i_o, we will then be able to write all the hyperreal numbers in decimal form.

The first part of this section is somewhat optional. There are three main reasons for studying it.
First, some students have trouble in believing in infinitesimals and other hyperreal numbers unless they can see these numbers and computations with them in a somewhat familiar concrete decimal form.
Second, it becomes clear the hyperreal numbers have the same arithmetic properties as the real numbers. Hyperreal arithmetic operations in decimal notation are only a step more difficult than those for the real numbers in unending decimal form.
Third, you should be able to explain to loved ones the interesting concepts you are learning in calculus!

I. The hyperreal numbers and their decimal representations

First, the real numbers in hyperreal form. They must have digits at infinite places so that they are in hyperreal form.

$4 = 4.000\cdots000{,}000\cdots$

$\frac{1}{2} = 0.500\cdots000{,}000\cdots$

$\frac{7}{3} = 2.333\cdots333{,}333\cdots$ Long division requires the 3's at infinite places also.

$\frac{3}{11} = 0.2727\cdots727{,}2727\cdots$ We chose N_o to be an *even* infinite integer.

$\pi = 3.14159\cdots???{,}???\cdots$ We don't know what all the digits are, but they exist.

There is a technical difference between a real number r and its hyperreal form, written r^*. For example

$$r = 2.666\cdots \neq r^* = 2.666\cdots666{,}666\cdots.$$

because r^* has digits at infinite places and r does not. (We say, "The *hyperreal extension* of the real number $r = 2.666\cdots$ is the hyperreal number $r^* = 2.666\cdots666{,}666\cdots$". However, we will not always show or say their distinction because in context we always know with which form we are dealing.)

There are three types of hyperreal numbers, categorized according to their relative sizes: the ***infinitesimals*** which we met in the previous section, the ***finite hyperreal numbers***, and the ***infinite hyperreal numbers***.

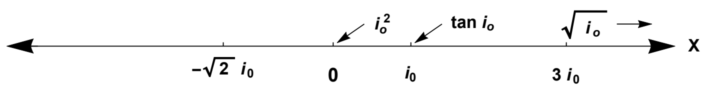
These three categories combined are called the set of hyperreal numbers $\mathbb{R}^*$. Our next task is to construct these numbers. We begin by reviewing the infinitesimals which we already familiar

1. The infinitesimals *An infinitesimal i is a hyperreal numbers with zeros at all finite decimal places*; so it is smaller in size than any nonzero real number. $0 = 0.000\cdots000{,}000\cdots$ is an infinitesimal and is the only infinitesimal that is (the hyperreal extension of) a real number. There are positive and negative infinitesimals.

Examples of infinitesimal hyperreal numbers

$$0 = 0.000\cdots000,000\cdots$$
$$i_o = 0.000\cdots001,000\cdots$$
$$3.1\,i_o = 0.000\cdots003,100\cdots$$
$$-\tfrac{1}{3}i_o = -\,0.000\cdots000,333\cdots$$
$$\sqrt{2}\;i_o = 0.000\cdots001,414\cdots$$
$$i_o^2 = 0.000\cdots000,000\cdots001,000\cdots$$
$$5692\,i_o + \sqrt{2}\;i_o = 0.000\cdots0005693,414\cdots$$

Near $x = 0$ there a family of infinitesimals which includes all multiples of i_o. The line near $x = 0$ is shown infinitely magnified by the amount 10^{N_o} (the arrowheads indicate infinitely magnified parts of the hyperreal line) in order to be able to see the infinitesimals. Note that i_o^2 is much smaller than i_o and that $\sqrt{i_o}$ is much larger than i_o.



2. Finite hyperreal numbers of the form $r^* + i$, $r \neq 0$ Every real number r (in hyperreal form) is surrounded by hyperreal numbers infinitesimally close to r. The general form of such finite hyperreal numbers is $h = r^* + i,\ r \neq 0$, where i is an infinitesimal. Below are a few of the hyperreal numbers infinitesimally close to 0. It is a homework exercise to write each explicitly in the form r^* + i. For example,

$$\tfrac{1}{3} + i \;=\; 0.333\cdots\begin{cases}\vdots\\ \cdots 331,000\cdots\\ \cdots 332,333\cdots\\ \cdots 333,333\cdots\\ \cdots 333,533\cdots\\ \cdots 334,333\cdots\\ \vdots\end{cases}$$

Important Observation **There are as many hyperreal numbers infinitesimally close to every real number r as there are in the set of all real numbers. This means you can do all analogs of real number algebra infinitesimally close to r. Understand this!**

Examples of finite hyperreal numbers

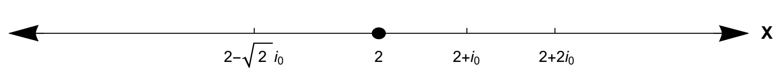
$$2 + i_o \;=\; 2.000\cdots001,000\cdots$$
$$\tfrac{1}{3} + i_o \;=\; 0.333\cdots334,333\cdots$$
$$2 - \tfrac{1}{3}i_o \;=\; 1.999\cdots999,666\cdots$$
$$17 + \sqrt{2}\;{i_o}^2 \;=\; 17.000\cdots000,000\cdots001,414\cdots$$

an irrational hyperreal number

The decimal expansions above are done by ordinary decimal calculations in hyperreal form. For example, for $2 + i_o$

$$\begin{array}{rl} & 2.000\cdots000,\,000\cdots\\ (+) & \underline{0.000\cdots001,\,000\cdots}\\ & 2.000\cdots001,\,000\cdots\end{array}$$

The infinitely magnified hyperreal line near $x = 2$ is shown below with a few nearby points plotted.



3. Infinite hyperreal numbers There are positive and negative infinite numbers. The reciprocal of an infinitesimal must be an infinitely large number. Our special infinite integer $\boldsymbol{I_o}$ is

$$I_o \equiv \frac{1}{i_o} = \frac{1}{10^{-N_o}} = 10^{N_o} = 1{,}000 \cdots 000$$

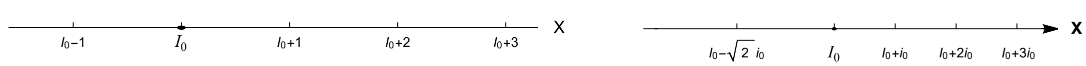
The 1 is in the $(N_o + 1)^{\text{th}}$ place as shown by the comma.

An *infinite hyperreal number* in decimal form has a digit at an infinite place to the left of the decimal point.

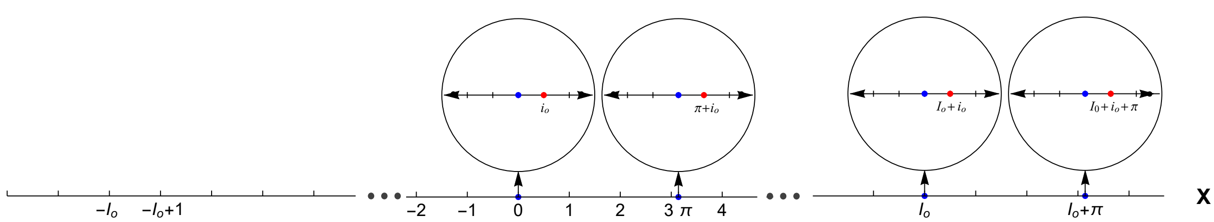
Examples of infinite hyperreal numbers

$I_0 + 5 = 1{,}000 \cdots 005$	infinite integer
$2 I_0^2 + \frac{1}{3} + 5 i_o = 2{,}000 \cdots 000{,}000 \cdots 000.333 \cdots 338, 333 \cdots$	infinite rational number
$-I_0 - \pi i_o = -1{,}000 \cdots 000.000 \cdots 003, 141 \cdots$	negative infinite irrational

Below is the hyperreal line near I_o as well as a infinite magnification of it.



As a summary of the hyperreal line, showing the three above categories of hyperreal numbers, see the figure below.



The hyperreal line is both longer than and finer than the real line!

Longer: You can explore the end behavior of functions at infinite values of space or time.

Finer: Surrounding the hyperreal form of every real number r, there is a continuum of points infinitesimally close to it. This allows you to explore computationally the behavior of a a function near r in great detail.

Algebra with the Hyperreal Numbers $\mathbb{R}^*$

It is clear that the hyperreal numbers, because they are decimal numbers, calculations with them work just like calculations with the real numbers. So the algebra of hyperreal numbers works just like that for the real numbers.

Numbers

One caution is that in theory when we are doing arithmetic or algebra with hyperreal numbers, all real numbers must in theory be written in their *hyperreal extension* form. For example,

$$\frac{1}{3} + i_o = 0.333\cdots + 0.000\cdots001,000\cdots$$

does not work because the decimal numbers have different lengths:

```
       0.333 · · · ???, ??? · · ·
(+)    0.000 · · · 001, 000 · · ·
       0.333 · · · ???,??? · · ·
```

But $\left(\frac{1}{3}\right)^* + i_o = 0.333\cdots333,333\cdots + 0.000\cdots001,000\cdots = 0.333\cdots334,333\cdots$ does work:

```
       0.333 · · · 333, 333 · · ·
(+)    0.000 · · · 001, 000 · · ·
       0.333 · · · 334, 333 · · ·
```

Formulas

All formulas or identities from real number algebra translate directly into hyperreal algebra formulas or identities.

$$(x + a)^2 = x^2 + 2\,a\,x + a^2$$

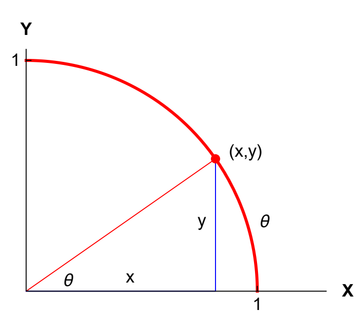
even if x and a are hyperreal numbers. This is because this hyperreal formula is derived using the same properties enjoyed by both the real and hyperreal numbers.

$$\sin^2\theta + \cos^2\theta = 1$$

even if θ is a hyperreal number. Let us show this. By the unit circle definitions of the trig functions, $\cos\theta = x$ and $\sin\theta = y$, where (x, y) is the point on the unit circle $x^2 + y^2 = 1$ at the end of the arc length θ. We define the trig functions in the same way whether the arc is described by real numbers or by hyperreal numbers. So if θ is a hyperreal number, so are $x = \cos\theta$ and $y = \sin\theta$ and therefore

$$x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$$

in this case also.



Sequences Unending hyperreal sequences behave just like those of unending real sequences in the sense that the same questions that can be asked and answered about the real sequences also apply to the hyperreal ones. Perhaps more surprising, this is also true for closed sequences.

Closed Sequence Principle Every mathematical question which can be answered for a finite sequence $r_1, r_2, \cdots, r_n$ can be answered for a closed infinite terminating sequence $h_1, h_2, \cdots, h_N$.

For example

2^{-4} is the least element of the finite sequence $\{1, 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}\}$.

2^{N_o} is the greatest element of the infinite sequence $\{1, 2^1, 2^2, \cdots, 2^{N_o}\}$.

But this cannot always be done for a non–terminating sequence:

$\{1, 2^{-1}, 2^{-2}, 2^{-3}, \cdots\}$ does not have a least element!

Functions A ***real function*** is one involving only explicit real numbers and variables. For example, $f(x) = x^2 + 3x + 5$ is a real function and

$f(2) = 2^2 + 3 \cdot 2 + 5 = 15$

a real number.

In calculus we will often want to explore a function such as f infinitesimally close to a real number r. To do this we will compute $f(r^{*}+i)$ for every infinitesimal i. In order to do this we must translate f into a hyperreal function so that x can be a hyperreal number; to do this, the real numbers in f must, in theory, be written as hyperreal numbers. We write

$f^{*}(x) = x^2 + (3.000 \cdots 000,000 \cdots)x + 5.000 \cdots 000,000 \cdots$

We normally do not show this *hyperreal extension* f* of the real function f explicitly. Such a hyperreal function of a hyperreal number is a hyperreal number. For example

$f(2+3i_o) = (2 + 3i_o)^2 + 3(2 + 3i_o) + 5$

$= 4 + 12i_o + 9{i_o}^2 + 6 + 9i_o + 5$

$= 15 + 21i_o + 9{i_o}^2$

$= 15.000 \cdots 021,000 \cdots 009,000 \cdots$.

All the real functions of a real variable we will use have a hyperreal extension.

An example of an ***explicit hyperreal function*** which is not the hyperreal extension of a real function is $g(x) = 2x + 3i_o$.

$g(2) = 2 \cdot 2 + 3i_o$

$= 4.000 \cdots 003,000 \cdots$

a hyperreal number. We will not need such functions in this course and only once, perhaps, in the next course.

Exercises

1. Write each hyperreal number in both fractional and decimal form. Use commas appropriately.

   a. $\frac{1}{8}$ b. $\frac{1}{9}$ c. $\frac{1}{9}i_o$

   d. $2+3i_o$ e. $2-3i_o$ f. $5I_0+71+5i_o$

2. Write each in decimal form.

   a. $3\,i_o+2\,i_o$ b. $(3\ \ +2\,i_o\,)^2$ c. $\frac{2+3\,i_o}{5}$ d. $3I_o^2+5\,I_o+7$

3. Show by hyperreal long division that $\frac{7}{3}=2.333\cdots333,333\cdots$.

4. A positive *hyper-infinitesimal* is a number smaller in size than any positive infinitesimal. Starting with the hyper-infinitesimals, the infinitesimals and the real numbers, you can construct the hyper-hyperreal numbers and the hyper-hyperreal number line. Invent a decimal representation for the hyper-hyperreals. Give a few examples. (You will see in this course that infinitesimals are sufficient for doing the calculus of real–valued functions, but in the next course that hyper-infinitesimals are required for the calculus of hyperreal valued functions.)

5. Find the hyperreal extension of each real function.

   a. a. $f(x)=x^3-\frac{1}{3}x+5$

   b. $g(\theta)=\tan\theta$

   c. $h(x)=2^x$ Hint: recall how you define 2^x for x a real irrational number.

Solutions

1. a. $0.1250\cdots000,000\cdots=\frac{1}{8}$

   c. $0.000\cdots000,111\cdots=\frac{1}{9,000\cdots000}$

   e. $1.999\cdots997,000\cdots=\frac{1,999\cdots997}{1,000\cdots000}$

2. b. $9.000\cdots012,000\cdots004,000\cdots$ d. $3,000\cdots005,000\cdots007$

3.

$$3\,\overline{)\,7.000\cdots000,000\cdots}\quad \text{quotient: } 2.333\cdots333,333\cdots$$

5. a. It is OK as written with the understanding that 1/3 and 5 are understood as hyperreal numbers.

   b. It is OK as written with the understanding that θ is a hyperreal angle.

   c. $2^{\frac{p}{q}}$ makes sense even if p and q are hyperreal integers.

$$2^{\frac{p}{q}}=\sqrt[q]{2^p}$$

Find a sequence of hyper-rational numbers approaching x and then compute the sequence composed of raising each element as a power of 2. This sequence approaches 2^x.

0.6 At the End of a Hyperreal Calculation

We will want the hyperreal numbers in order to do some `hyper-precise' calculations with the hyperreal extensions of real functions. However, at the end of a calculation a real number is all the precision we need.

So for an infinitesimal answer we drop the digits at infinite places; 'Every infinitesimal *rounds off* to 0'. For a finite hyperreal number answer $r^* + i$, we also drop the infinitesimal part, *rounding off* to the nearest real number r. We do not distinguish between positive infinite answers (to us mortals, all positive infinite numbers are just equally incredibility large); we say that any positive infinite number rounds off to *plus infinity*, written $+\infty$, and likewise negative infinite numbers round off to $-\infty$.

You now should have a clear precise understanding of the meaning and use of the symbol ∞.

Rules for Rounding Off **The symbol ≈> denotes rounding off.**

1. **Infinitesimals** — **i ≈> 0**
2. **Finite hyperreal numbers** — **$h = r^* + i$ ≈> r**
3. **Infinite numbers** — **I ≈> $+\infty$, $-$I ≈> $-\infty$ (I positive infinite)**

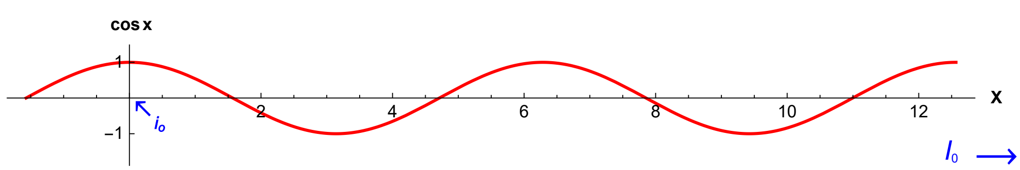
Examples Rounding off is often easy.

$3i_o$ ≈> 0 — 17.250 ⋯000,000 ⋯ ≈> $17\frac{1}{4}$

$5 + 2i_o + 7i_o^2$ ≈> 5 — 6.781 ⋯ 034, 172 ⋯ ≈> 6.781 ⋯

$7I_0 - 84$ ≈> $+\infty$ — $\cos i_o$ ≈> 1

0.333 ⋯ 33, 433 ⋯ ≈> 0.333 ⋯ = $\frac{1}{3}$ — $\cos I_0$ exists, but do not know its value.



If you do not prefer the phrase 'rounds off to' related to the optional understanding of the hyperreals as decimals, you can use for ≈> by saying '***associates with*** or '***associates with the extended real number***'

Examples Let x be a real number and dx a positive infinitesimal

x + dx 'associates with' x

$\frac{1}{dx}$ 'associates with' $+\infty$

We have a great symbol, ≈> , for 'associates with'.
Need a better phrase for 'associates with'.
GRAND PRIZE. One free PDF of this book!

Example The following hyperreal numbers associate with the same real number.

$$7.333\cdots \begin{cases} \vdots \\ \cdots 331,000\cdots \\ \cdots 332,333\cdots \\ \cdots 333,333\cdots \\ \cdots 333,533\cdots \\ \cdots 334,567\cdots \\ \end{cases} \approx> 7.333\cdots = \frac{22}{3}.$$

The Extended Real Numbers These numbers are widely used in answers in pure and applied mathematics. Their meaning is clear in the context of associating hyperreal numbers with a real number. Verifying the extended real arithmetic facts is left as an exercise.

The *extended real numbers* are the real numbers plus the symbols $+\infty$ and $-\infty$.

The result of finishing any hyperreal calculation can only be one of:

1. **a real number r**
2. $+\infty$ **or** $+\infty$
3. **does not exist.**

When we do calculus, we will agree that an answer can only be an *extended real number* or that the answer *does not exist*.

Extended Real Arithmetic Facts For r a real number

$r + (+\infty) = +\infty \qquad r + (-\infty) = -\infty$

$r \cdot (+\infty) = +\infty, \; r > 0 \qquad r \cdot (+\infty) = -\infty, \; r < 0$

$+\infty + (+\infty) = +\infty \qquad -\infty + (-\infty) = -\infty$

The early users of calculus often used infinitesimals and other hyperreal numbers in their theory and calculations (calculus then was often called ***infinitesimal calculus***). In fact, just about all the formulas and techniques of calculus you are likely to meet were discovered using infinitesimals. Infinitesimals came under suspicion because no one understood them in a rigorous way or even had a confident intuition about them (decimal numbers were were only beginning to be used in the seventeenth century). By the twentieth century, mathematicians stopped using infinitesimals and used Weierstrass' rigorous, but difficult, ϵ–δ method instead. Even so, most scientists and engineers continued using infinitesimals because of their intuitive appeal and the way they simplified derivations and calculations. The theory of hyperreal numbers was put on a rigorous foundation in the mid–twentieth century. However, that rigorous treatment is too difficult and tedious for most calculus beginners.

Nevertheless, despite our very elementary introduction to the hyperreal numbers, your knowledge of the hyperreal number system now should be as complete and intuitively understood as your knowledge of the real number system, and furthermore, because mathematicians have given the hyperreals their official endorsement, you can use them with confidence.

Final note We will normally only use the hyperreal numbers symbolically. That is, we will write our finite hyperreal numbers in the form $x + \mathrm{dx}$ where x is *a* real number in hyperreal form and dx is an infinitesimal. We will have no need to work with their decimal form (we emphasized their decimal form so you would feel comfortable with them and help recognize they have the same algebraic properties as the real numbers).

You will never see hyperreal numbers in decimal form again!

A Short Axiomatic Summary of the Hyperreal Number System

Pure mathematicians, when describing a mathematical system, state its ***definitions*** and list its ***axioms*** (statements taken to be true) and then derive from them ***theorems*** (true statements about the system). For beginners it is usually better to develop a good intuition about the system as we did in the previous two sections for the hyperreal number system. However, once you understand the hyperreal system, the following provides a quick summary and review.

Definition An ***infinitesimal*** dx is a number smaller in size than every positive real number x.

Axiom Infinitesimals exist. (About 1969 Abraham Robinson proved infinitesimals exist.)

Axiom The ***hyperreal numbers***, consisting of all algebraic combinations of the real numbers (in hyperreal form) and infinitesimals, satisfy the usual laws of the real numbers.

Examples

$2dx + 5\,dx^2 = dx(2 + 5\,dx)$

$(3 + dx)^2 = 9 + 6\,dx + dx^2$

$\frac{1 + 2\,dx}{dx} = \frac{1}{dx} + 2, \; dx \neq 0$

Definition ≈> associates hyperreal numbers with extended real numbers:

1. infinitesimals dx ≈> 0
2. finite numbers x + dx ≈> x
3. infinite numbers (X positive}

   X ≈> +∞

   -X ≈> -∞

At the end of a hyperreal calculation, we want an extended real number.

Examples

$2dx + 5\,dx^2$ ≈> 0

$(3 + dx)^2 = 9 + 6\,dx + dx^2$ ≈> 9

$\frac{1}{dx}$ ≈> +∞, $dx > 0$.

Definition Two hyperreal numbers h_1 and h_2 are ***asymptotically equal***, written $h_1 \approx h_2$ if

$\frac{h_1}{h_2} = 1 + \epsilon$ where ϵ is an infinitesimal.

Theorem $A \approx B, C \approx D \iff$

1. $A\,C \approx B\,D$
2. $\frac{A}{C} \approx \frac{B}{D}$

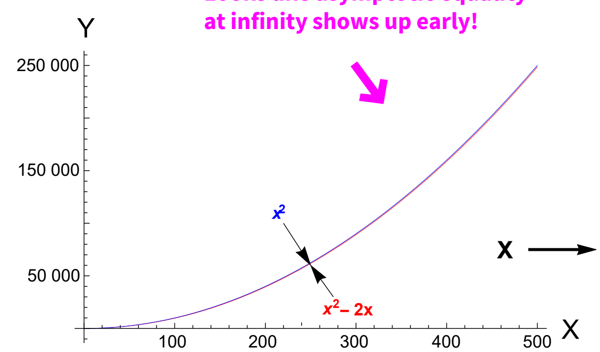
Looks like asymptotic equality at infinity shows up early!

Examples

$2dx + 5\,dx^2 \approx 2dx$ ⟶ **Proof** $\frac{2dx + 5dx^2}{2dx} = 1 + \frac{2}{5}dx = 1 + \epsilon$

$(3 + dx)^2 = 9 + 6\,dx + dx^2 \approx 9$

$X^2 - 2X \approx X^2$, X an infinite hyperreal number. ⟶



Exercises Semi-memorize the **Axiomatic Summary**.

Note: **≈ allows us to make simplifications while doing calculations. It is more flexible than = but in the end results in the same extended real answer!**

1. Provide five different ≈ answers for each.

   a. $3dx - dx^3$
   b. $7 - 4\,dx$
   c. $2X - 4X^2 + 7$

2. Round off each

   a. $7.333\cdots333{,}733\cdots$
   b. $0.000\cdots012{,}345\cdots$
   c. $\frac{I_0}{15}$
   d. $\sin i_o$
   e. $\sin I_0$
   f. $\sqrt[I_o]{2}$
   g. $\tan\left(\frac{\pi}{2} - i_o\right)$, $\tan\left(\frac{\pi}{2} + i_o\right)$, $\tan\left(\frac{\pi}{2}\right)$
   h. $\sin(\pi I_0)$
   i. $i_o^{\,i_o}$

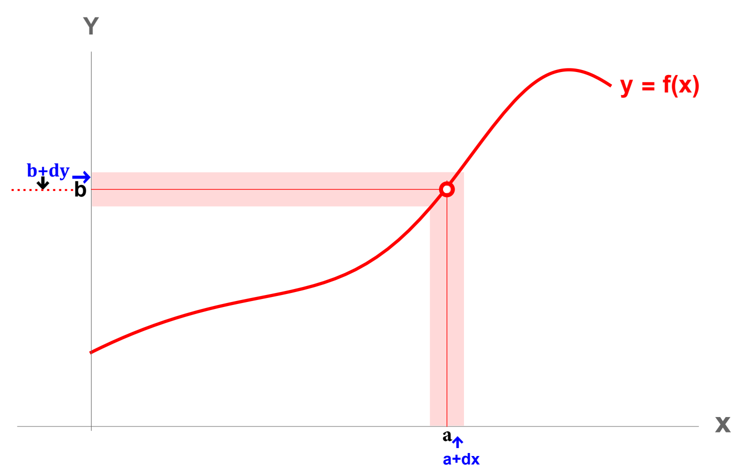
3. For each expression:

   First simplify by hyperreal algebra (assume dx is not 0)
   Then round off, taking dx = 0).

   a. $\frac{(1+dx)^2-1}{dx}$
   b. $\frac{(x+dx)^2-x^2}{dx}$
   c. $\frac{1}{dx}\left(\frac{1}{1+dx} - 1\right)$
   d. $\frac{1}{dx}\left(\frac{1}{x+dx} - \frac{1}{x}\right)$
   e. $\frac{\sqrt{1+dx}-1}{dx}$
   f. $\frac{\sqrt{x+dx}-x}{dx}$
   g. $\frac{\sqrt[3]{dx}}{dx}$
   h. $\frac{\sqrt{dx}}{dx}$

4. The problems in #3 are actual calculus problems. (Don't worry what they mean now.) See if you can relate them to the process illustrated in the diagram from the front cover

   In each part identify f(x) and a.
   What is the problem if dx = 0?



5. For each expression:

   X is a positive infinite number.
   First simplify by hyperreal algebra if required.
   Then round off.

   a. $\frac{X}{X+1}$
   b. $\frac{X^2}{2X+1}$
   c. $\frac{X^3}{3X^2+2X+1}$
   d. $\frac{X}{X^3-5X^2}$
   e. $\frac{X+dx}{X-dx}$
   f. $\frac{\sqrt{X+5}-x}{X^2}$
   g. $\frac{\sqrt{1-X}}{X+5}$
   h. $\sqrt{X^2+7} - X$

Solutions

1. b. $7 - 4\,dx \approx 7 - 4\,dx$

$7 - 4\,dx \approx 7$

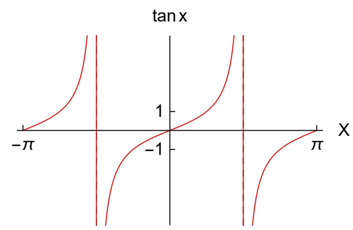
$7 - 4\,dx \approx 7 + 24\,dx$

$7 - 4\,dx \approx 7 - 24\,dx$

$7 - 4\,dx \approx 7 - 24\,dx + dx^2$

2. a. 7.333 ⋯ b. 0 c. +∞
   d. 0 e.* f. 1
   h. 0 g. +∞, -∞, DNE

* Walk to I_o and have a look.



4e. a = 1 won't work (why?). Go to 1 + dx instead.

3 e. $\dfrac{\sqrt{1+dx} - 1}{dx}$

$= \dfrac{\sqrt{1+dx} - 1}{dx}\,\dfrac{\sqrt{1+dx} + 1}{\sqrt{1+dx} + 1}$ rationalize

$= \dfrac{(1+dx) - 1}{dx(\sqrt{1+dx} + 1)}$

$= \dfrac{dx}{dx(\sqrt{1+dx} + 1)}$

$= \dfrac{1}{\sqrt{1+dx} + 1)}$ simplified

$\approx> \dfrac{1}{2}$ rounded off

at b +dy

at $b = \frac{1}{2}$.

5 a. $\dfrac{X}{X+1}$

$= \dfrac{1}{1 + 1/X}$

$\approx \dfrac{1}{1+0}$

$\approx> 1$

5 b. +∞

c. +∞

5 d. 0

e. 1

g. DNE

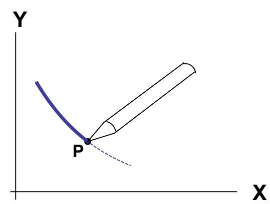
Chapter 1 Continuity and Limits

Limits is the new computation we need to do calculus. In fact, any mathematics using limits is called calculus. We begin with the the study of continuity, a topic about which you have some intuition. Once you have a precise understanding of continuity, limits will be easy.

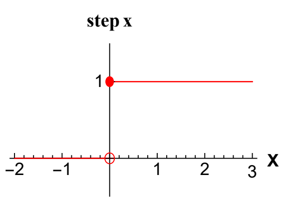
1.1 Continuity

Introduction

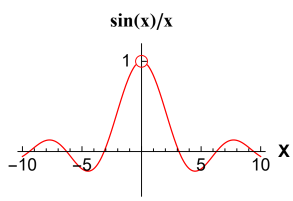
Your intuitive understanding of continuity may be something like this: "A function is continuous at a point **P** if you can draw its curve *through* the point without lifting (or putting down) your pencil there". This understanding comes from the ordinary literal meaning of the word continuous, namely not having any breaks.



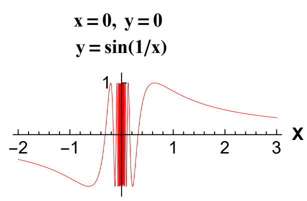
However, it would not be a good, generally applicable definition of continuity; this definition works quite well for jump discontinuities as illustrated on the first graph below. For other types of continuity the 'without lifting your pencil' definition often is inadequate (see the other graphs below). We clearly need a definition of continuity which is stated in precise mathematical language and which works for all functions.



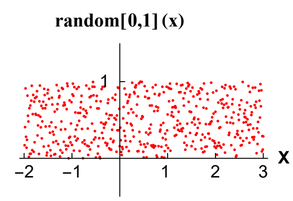
Jump discontinuity
Clearly not continuous at x = 0.



Hole discontinuity
Lift your pencil zero distance at x = 0?



Oscillatory discontinuity
What happens at x = 0? Not clear.

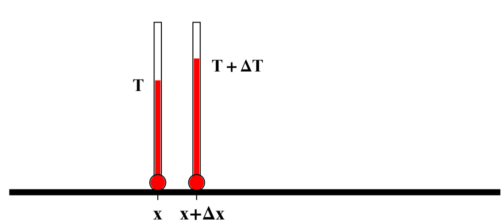


Scattered values discontinuity
Not continuous anywhere.

As a clue for a good definition of continuity, let us look at what continuity means in science. The concept of continuity is important for science as well as the theory of calculus. Without continuity it would be impossible to do measurements in science. As an example, consider the problem of measuring the temperature T of a rod at a point x. It is impossible to place a thermometer *exactly** at the point x. Instead, despite our best human efforts, we find it placed at the point $x + \Delta x$, where Δx is the error in placement. Associated with this Δx there will be an error ΔT in the temperature T we wish to read. Hopefully, if Δx is small, ΔT will also be small; otherwise we could have no confidence in our measurement T.

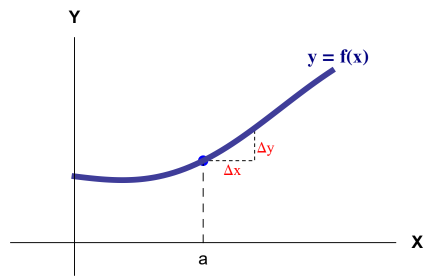
This expectation is called the ***continuity*** of the temperature function.

* Shaky hand
Imperfect vision
Quantum mechanics



The hyperreal definition of Continuity

The temperature example suggests that we define the continuity of a function $y = f(x)$ at $x = a$ to mean that if Δx is small, then Δy will also will be small. Fortunately we have an unambiguous definition of small, namely a number is ***small*** if it is an ***infinitesimal***.

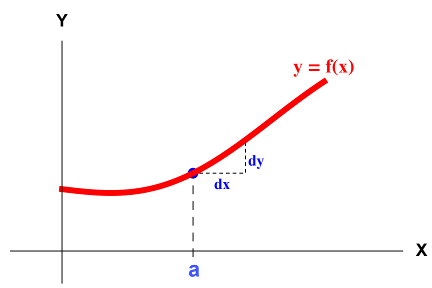


We will use the symbols dx and dy for infinitesimals when doing calculus hyperreal calculations. As you would expect, dx means an infinitesimal change in x in comparison with Δx which means a real number change in x.

The definition of continuity and and the proofs of the continuity of some familiar functions and the proofs of continuity theorems is the legitimate beginning of serious calculus. Make sure you master continuity.

Definition of Continuity f is continuous at (the real number) $x = a$ means

1. $f(a) = b$ exists.
2. For every infinitesimal dx, $dy = f(a+dx) - f(a)$ is an infinitesimal.



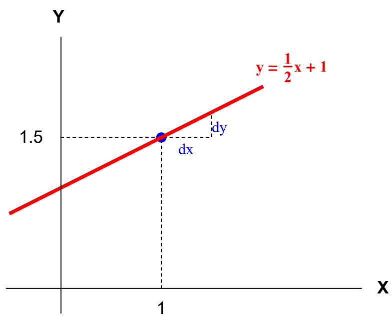
Two numbers are said to be ***infinitesimally close*** if their difference is an infinitesimal. So the above precise definition means that f is continuous at $x = a$ if whenever x is infinitesimally close to a, y is infinitesimally close to the value f(a). We also require f(a) to exist so there is no ‘hole’ in the graph at $x = a$. For convenience, on a graph we usually show infinitesimals such as dx and dy improperly as small finite numbers rather than as infinitesimals, which would require an infinite magnification of the axes to see.

Proving the continuity of functions using the definition

Let us begin by proving the continuity of the examples which range from easy ones to harder ones.

Example 1 Prove that $f(x) = \frac{1}{2}x + 1$ is continuous at $x = 1$.

dx and dy shown infinitely magnified



Proof

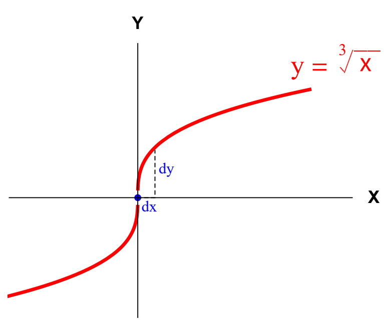
1. $f(1) = \frac{3}{2}$, exists
2. Let dx be any infinitesimal. Then

$$dy = f(1+dx) - f(1)$$
$$= \tfrac{1}{2}(1+dx) + (1 - \tfrac{3}{2})$$
$$= \tfrac{1}{2}dx,$$

an infinitesimal. half of an infinitesimal is an infinitesimal

End of Proof

Example 2 Show that $f(x) = \sqrt[3]{x}$ is continuous at $x = 0$.



Proof

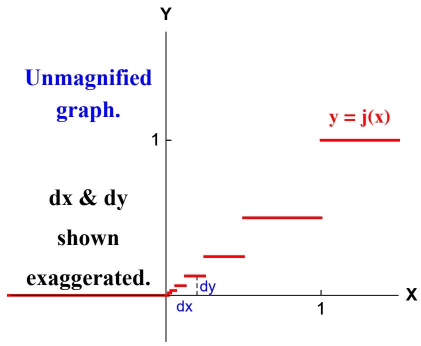
1. $f(0) = \sqrt[3]{0} = 0$, exists
2. Let dx be any infinitesimal. Then

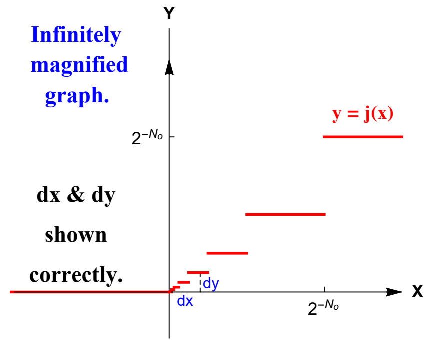
$$dy = f(0+dx) - f(0)$$
$$= \sqrt[3]{0 + dx} - 0$$
$$= \sqrt[3]{dx}\,,$$

an infinitesimal. (the cube root of a small number is a small number)

End of Proof

Example 3 Show that $j(x) = \begin{cases} 0, & x \le 0 \\ \frac{1}{2^n}, & \frac{1}{2^n} < x \le \frac{1}{2^{n-1}},\ n = 1, 2, 3, \cdots \\ 2^n, & 2^{n-1} < x \le 2^n,\ n = 1, 2, 3, \cdots \end{cases}$ is continuous at $x = 0$.





Proof

1. $j(0) = 0$, exists
2. If dx is any negative infinitesimal. Then

   $dy = j(0+dx) - j(0)$
   $= 0 - 0$
   $= 0$,
   an infinitesimal.

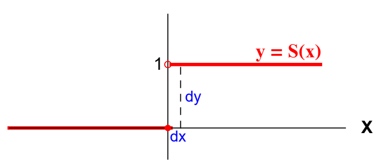
   If dx is a positive infinitesimal, then

   $dy = j(0+dx) - j(0) = j(dx)$,
   an infinitesimal between 0 and dx.

End of Proof

If we examine the above curve infinitely magnified about the origin, it still looks exactly like the original curve near the origin. It clearly is not `hypercontinuous' at $x = 0$ because there are infinitesimal sized jumps just to the right of the origin. However, we can draw it through the origin any infinitesimal amount without lifting our pencil a *real amount*. What really matters is that if dx is an infinitesimal, dy is an infinitesimal. Be clear about this.

Example 4 Show that the unit step function $S(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$ is not continuous at $x = 0$.



Proof

1. $S(0) = 0$, exists
2. Let $dx > 0$ be an infinitesimal. Then

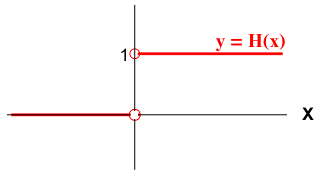
   $dy = S(0+dx) - S(0)$
   $= 1 - 0$
   $= 1$,
   not an infinitesimal.

   So S is not continuous at $x = 0$.

End of Proof

Example 5 Prove that the Heaviside function $H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$ is not continuous at $x = 0$.

Proof



1. $H(0)$ does not exist. So H is not continuous at $x = 0$.

End of Proof

In the first three examples it was clear that if dx is an infinitesimal, then so was $\frac{1}{2}dx$, $\sqrt[3]{dx}$, and a number smaller in size than dx. In more complicated problems it is useful to have a theorem which helps us spot immediately when dy is an infinitesimal.

Relative Size Theorem Let (with or without subscripts) i be a positive infinitesimal, h be a positive finite hyperreal number, and I be a positive infinite hyperreal number. Then

1. The following are infinitesimals

   $i_1 \pm i_2$ (frequently used)

   $i_1 \cdot i_2$

   $h \cdot i$ (frequently used)

   $\frac{i}{h}$

   $\frac{h}{I}$

   i^n, n a positive integer

   $\sqrt[n]{i}$, n a positive integer

These in red are most frequently used in beginning calculus.

2. The following are finite hyperreal numbers

   $h_1 + h_2$ (also $h_1 - h_2$ unless h_1 and h_2 are infinitesimally close)

   $h_1 \cdot h_2$

   $\frac{h_1}{h_2}$

   $h \pm i$

3. The following are infinite hyperreal numbers

   $I_1 + I_2$

   $I_1 \cdot I_2$

   $h \cdot I$

   $\frac{I}{h}$

4. The following are ***indeterminate*** forms; this means that examples can be given for each where the result could be more than one of an infinitesimal, a finite hyperreal, or an infinite number.

   $\frac{i_1}{i_2}$

   $I_1 - I_2$

   $i \cdot I$

   $\frac{I_1}{I_2}$

If the h or i or I's are negative, the results of this theorem are readily modified.

We shall not prove much of this theorem because the results are rather intuitive. For example, $h\cdot i$ is an infinitesimal. Intuitively this says that a medium sized number times a small number is a small number, e.g., '$2 \times 0.001 = 0.002$' or $(275 + 2i_o)i_o = 0.000\cdots00275{,}000\cdots002{,}000\cdots$. An elementary proof is the observation that multiplying an infinitesimal which has zeros at all finite decimal places by a finite number results in a number with zeros at all finite places, an infinitesimal. A more formal proof would be to prove $|h\,i|$ is smaller than any positive real number.

$I_1 - I_2$ is indeterminate because, for example, $\mathbf{I} - 2\mathbf{I}$ is negative infinite, $\mathbf{I} - \mathbf{I} = 0$, and $3\mathbf{I} - \mathbf{I}$ is positive infinite. You should do examples to illustrate some of the others.

In the following examples we use the above theorem to determine whether dy is an infinitesimal. We also will determine the continuity at any suitable domain value x rather than only at a given point $x = a$; it usually is not much more difficult to do so. x is understood as the hyperreal form of the real number x.

Example 6 A polynomial function Prove that $f(x) = x^2 - 3x + 3$ is continuous for all x.

Proof .

NOTE These Grade 10 type calculations are about as difficult as the algebra gets in this course. The corresponding epsilon-delta limit calculations are so difficult few students understand them and so fail to understand fully much of the calculus.

1. $f(x) = x^2 - 3x + 3$, exists.
2. $dy = f(x+dx) - f(x)$

$= ((x + dx)^2 - 3(x+dx) + 3) - (x^2 - 3x + 3)$

$= x^2 + 2x\,dx + dx^2 - 3x - 3dx + 3 - x^2 + 3x - 3$

$= 2x\,dx - 3dx + dx^2$

$= (2x - 3 + dx)\,dx,$ type $h\cdot i$ or, if $x = \frac{3}{2}$, $i_1 \cdot i_2$

an infinitesimal. **End of Proof**

Example 7 A rational function Prove that $f(x) = \frac{1}{x}$ is continuous for all $x \neq 0$.

Proof

1. $f(x) = \frac{1}{x}$, exists for $x \neq 0$.
2. Let dx be any infinitesimal. Then

$dy = f(x+dx) - f(x)$

$= \frac{1}{x+dx} - \frac{1}{x}$

$= \frac{x-(x+dx)}{x(x+dx)}$ get a common denominator

$= \frac{-dx}{x(x+dx)}$ type $h\,i$ since x is not the real number 0.

$= -\frac{1}{x(x+dx)}dx,$ Note also that if $x \neq 0$, then $x + dx \neq 0$.

an infinitesimal. **End of Proof**

Example 8 An algebraic function Prove that $f(x) = \sqrt{x}$ is continuous for all $x > 0$.

Proof

1. $f(x) = \sqrt{x}$, exists for $x > 0$.
2. Let dx be any infinitesimal. Then

$$dy = f(x+dx) - f(x)$$

$$= \sqrt{x + dx} - \sqrt{x}$$

$$= \frac{\sqrt{x+dx} - \sqrt{x}}{1} \cdot \frac{\sqrt{x+dx} + \sqrt{x}}{\sqrt{x+dx} + \sqrt{x}} \qquad \text{rationalizing the numerator}$$

$$= \frac{x+dx}{\sqrt{x+dx} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+dx} + \sqrt{x}}dx, \qquad \text{type h i since } x > 0$$

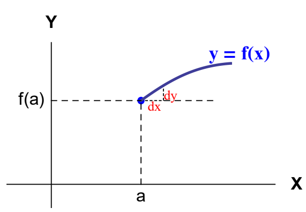
an infinitesimal.

End of Proof

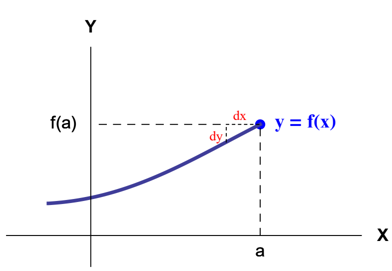
One-Sided Continuity

A function is not continuous at an endpoint of a domain interval because dx either cannot be positive or cannot be negative to the left or right of a point $x = a$. Nevertheless, it may be meaningful to talk about one–sided continuity there because you can start or stop with your pencil down. Also at points of discontinuity, the concept of one–sided continuity may be meaningful.

Definition A function f is ***continuous from the right*** at $x = a$ means
1. $f(a) = b$ exists.
2. For every infinitesimal $dx > 0$, $dy = f(a+dx) - f(a)$ is an infinitesimal.

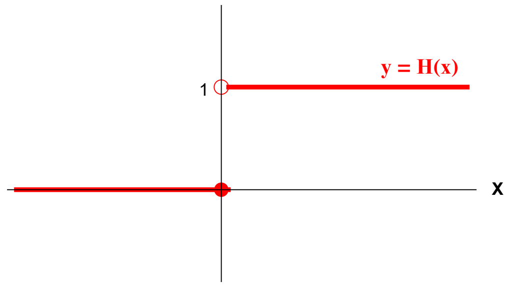


Definition A function f is ***continuous from the left*** at $x = a$ means
1. $f(a) = b$ exists.
2. For every infinitesimal $dx < 0$, $dy = f(a+dx) - f(a)$ is an infinitesimal.



Theorem f is continuous both from the left and the right at $x = a$ means f is continuous at $x = a$.

Example 9 Use the above theorem to determine graphically the continuity of the unit step function $S(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$.



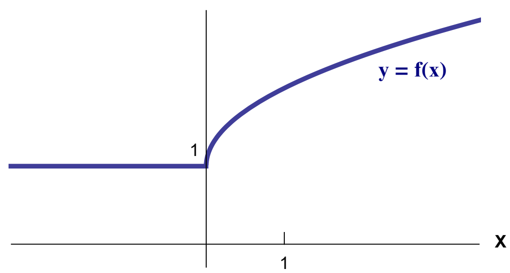
By inspection:

S(x) is continuous from the left at $x = 0$.

S(x) is not continuous from the right at $x = 0$.

Therefore S(x) is not continuous at $x = 0$.

Example 10 Determine graphically the continuity of the function $f(x) = \begin{cases} 1, & x \le 0 \\ 1 + \sqrt{x}, & x > 0 \end{cases}$.



By inspection:

f is continuous from the left at $x = 0$.

f is continuous from the right at $x = 0$.

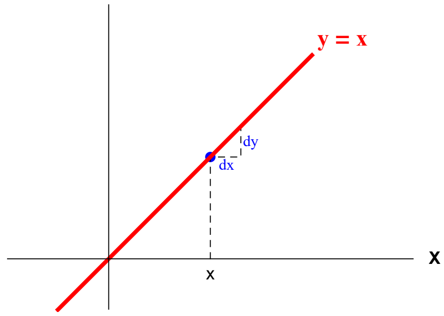
So f is continuous at $x = 0$.

We could of course have proved the two previous examples analytically

Basic Continuous Functions Theorems Here we list of some basic functions that are continuous in preparation for the next section where we prove the continuity of whole classes of functions. We will only prove the second one and give a graphical understanding of and an analytic proof of the fourth. The first and fifth ones are left as exercises. We also saw how root functions are proved continuous in the examples and exercises.

1. $f(x) = c$ is continuous for every $x = a$.
2. $f(x) = x$ is continuous for every $x = a$.
3. $f(x) = \sqrt[n]{x}$ is continuous for every $x = a$ if n is odd and for every $x = a > 0$ if n is even.
4. $f(x) = \sin x$ is continuous for every $x = a$.
5. $f(x) = \cos x$ is continuous for every $x = a$.

Proof of 2



1. f(x) = x, exists
2. Given any infinitesimal dx.

   dy = f(x+dx) – f(x)

   = (x + dx) – x

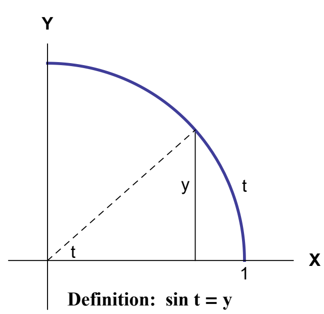
   = dx,

   an infinitesimal.

End of Proof

Proof of 4

In this proof we need the geometrically motivated definition of the sine function you learned in high school: sin t = y where t is the arc length of the unit circle as shown. Note that in this problem t, not x, is the independent variable. Also, we will show infinitesimal quantities as not very small real lengths.



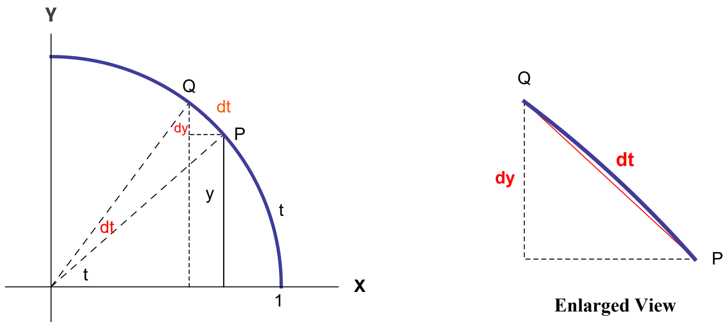
1. y = sin t, exists. — see drawing above for the unit circle definition of sin t
2. Given any infinitesimal dt.

   dy = sin(t + dt) – sin t — see drawing below

   $\leq \overline{PQ}$ — because the altitude of a right triangle is smaller than its hypotenuse

   ≤ dt, — because the line segment $\overline{PQ}$ is the shortest curve joining P and Q

   an infinitesimal.



End of Proof

In conclusion, using the hyperreal definition of continuity is often quite easy, involving only elementary algebra skills and the ability to spot an infinitesimal quantity quickly using the Relative Size Theorem.

Exercises In exercises 1 to 8 use our precise hyperreal definition of continuity.

#1. Prove that $f(x) = x^2$ is continuous at $x = 2$.

#2. Prove that $f(x) = x^2 - 2x + 1$ is continuous at $x = 3$.

#3. Prove that $f(x) = x^3 - 3$ is continuous at $x = 2$. Hint: Use $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

#4. Prove that $f(x) = \frac{1}{x}$ is continuous at $x = 1$.

#5. Prove that $f(x) = \frac{1}{2x - 1}$ is continuous at all $x \neq \frac{1}{2}$.

#6. Prove that $f(x) = \frac{x}{x+1}$ is continuous for all $x \neq -1$.

#7. Prove that $g(x) = \sqrt{2x}$ is continuous for all $x > 0$.

#8. Prove that the function $cr(x) = \sqrt[3]{x}$ is continuous for all $x \neq 0$.
Hint: use $(A - B)(A^2 + AB + B^2) = A^3 - B^3$ to rationalize the numerator.

#9. Prove the theorem, Basic Continuous Functions #1.

#10. Prove the theorem, Basic Continuous Functions #5.

#11. Prove that $f(x) = \sqrt{x}$ is continuous from the right at $x = 0$.

#12. Prove the continuity of each of the following more difficult functions.

a. $f(x) = x^4$ at for all x. Hint: $(A + B)^4 = A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4$

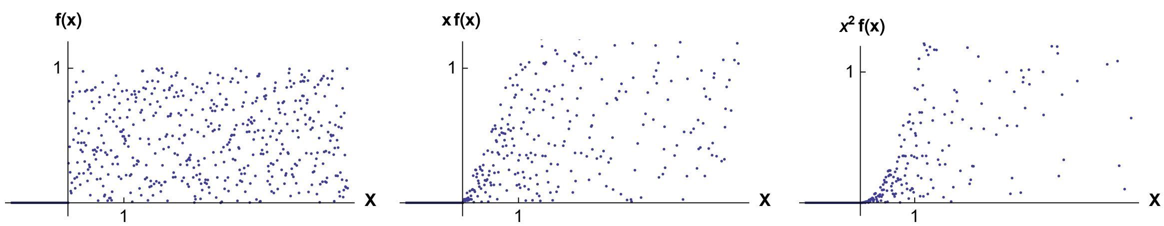
b. $g(x) = \frac{x^2}{x+1}$ for all $x \neq -1$

c. $h(x) = \sqrt{x^2 + 1}$ at $x = 3$

#13. a. Understand why $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at $x = 0$. Verify by graphing.

b. Prove that $g(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at $x = 0$. Verify by graphing.

#14. The function f(x) below has randomly produced values between 0 and 1 for $x > 0$; $f(x) = 0$ for $x \leq 0$. Which of the functions are continuous at $x = 0$?



#15. Show by example that $\frac{i_1}{i_2}$ is indeterminate.

#16. a. Use the hyperreal calculator on the website www.lightandmatter.com/calc/inf to explore the continuity of $y = 2^x$ at $x = 0$ taking $dx = d = i_o$ where d is the symbolic infinitesimal used in the calculator.

b. Use the result of part a to show that $y = 2^x$ appears continuous for all x.

#17. Give an example of a function which is continuous only at $x = 0$.

Solutions

#1. **Proof** $f(x) = x^2$ is continuous at $x = 2$.

1. $f(2) = 2^2 = 4$, exists.
2. Let dx be any infinitesimal.

$dy = f(2 + dx) - f(2)$
$= (2 + dx)^2 - 4$
$= (4 + 4\,dx + dx^2) - 4$
$= (4 + dx)dx$, type h i
an infinitesimal. **End of Proof**

#3. **Proof** $f(x) = x^3 - 3$ is continuous at $x = 2$.

1. $f(2) = 2^3 - 3 = 5$, exists.
2. Let dx be any infinitesimal.

$dy = f(2 + dx) - f(2)$
$= ((2 + dx)^3 - 3) - 5$
$= (8 + 12\,dx + 6\,dx^2 + dx^3 - 3) - 5$ expansion of $(A + B)^3$
$= (12 + 6\,dx + dx^2)dx$, type h i
an infinitesimal. **End of Proof**

#5. **Proof** $f(x) = \frac{1}{2x-1}$ is continuous for all $x \neq \frac{1}{2}$.

1. $f(x) = \frac{1}{2x-1}$, exists.
2. Let dx be any infinitesimal.

$dy = f(x + dx) - f(x)$

$= \frac{1}{2(x+dx)-1} - \frac{1}{2x-1}$

$= \frac{(2x-1)-(2(x+dx)-1)}{(2(x+dx)-1)(2x-1)}$

$= \frac{-2}{2(x+dx)-1)(2x-1)}dx$, type h i, $x \neq \frac{1}{2}$

an infinitesimal. **End of Proof**

#7. **Proof** $f(x) = \sqrt{2x}$ is continuous for all $x > 0$.

1. $f(x) = \sqrt{2x}$, exists for $x > 0$
2. Let dx be any infinitesimal.

$dy = f(x+dx) - f(x)$

$= \sqrt{2(x+dx)} - \sqrt{2x}$

$= \left(\sqrt{2(x+dx)} - \sqrt{2x}\right) \cdot \frac{\sqrt{2(x+dx)} + \sqrt{2x}}{\sqrt{2(x+dx)} + \sqrt{2x}}$ rationalizing the numerator

$= \frac{2(x+dx) - 2x}{\sqrt{2(x+dx)} + \sqrt{2x}}$

$= \frac{2}{\sqrt{2(1+dx)} + \sqrt{2}}dx$, type h i if $x > 0$

an infinitesimal.

End of Proof

#12 c. **Proof** $f(x) = \sqrt{x^2+1}$ for all x.

1. $f(x) = \sqrt{x^2+1}$, exists.
2. Let dx be any infinitesimal.

$dy = f(x+dx) - f(x)$

$= \sqrt{(x+dx)^2+1} - \sqrt{x^2+1}$

$= \left(\sqrt{(x+\Delta x)^2+1} - \sqrt{x^2+1}\right)\frac{\sqrt{(x+dx)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+dx)^2+1} + \sqrt{x^2+1}}$ rationalizing the numerator

$= \frac{((x+dx)^2+1) - (x^2+1)}{\sqrt{(x+dx)^2+1} + \sqrt{x^2+1}}$

$= \frac{2x\,dx + dx^2}{\sqrt{(x+dx)^2+1} + \sqrt{x^2+1}}$

$= \frac{2x + dx}{\sqrt{(x+dx)^2+1} + \sqrt{x^2+1}}dx$, type h i or $i_1\, i_2$ if $x = 0$

an infinitesimal. **End of Proof**

#13b. **Proof**

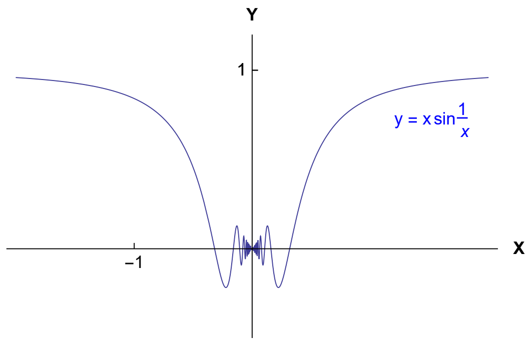
1. f(0) = 0, exists.
2. Let dx be any infinitesimal. then

   $dy = f(0+dx) - f(0)$

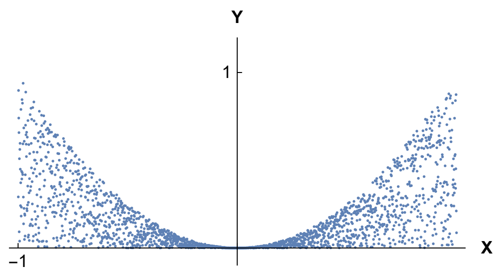
   $= dx \sin \frac{1}{dx}$, an infinitesimal times a number between -1 and 1

   an infinitesimal. Note the $dx = 0$ and $dx \neq 0$ cases.

End of Proof



#17.

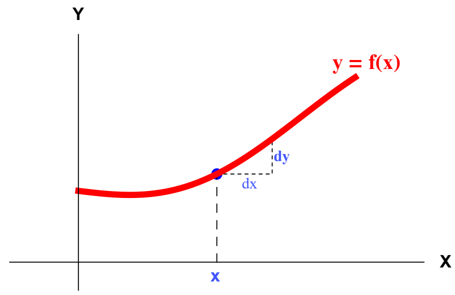


1.2 Getting Proficient at Determining Continuity

Using the definition of continuity is tedious. We will prove theorems that speed up determining the continuity of combinations of functions of known continuity.

Recall f is ***continuous*** at (the real number) x means

1. f(x) exists.
2. For every infinitesimal dx, dy = f(x+dx) – f(x) is an infinitesimal.



Continuity Theorems

The following set of theorems was introduced in the previous section.

Basic Continuity Theorems

1. f(x) = c is continuous for every x.
2. f(x) = x is continuous for every x.
3. f(x) = $\sqrt[n]{x}$ is continuous for every x if n is odd and for every x > 0 if n is even.
4. f(x) = sin x is continuous for every x.
5. f(x) = cos x is continuous for every x.

The next set of theorems apply to any continuous functions f and g.

General Continuity Theorems Let y_1= f(x) and y_2 = g(x) be continuous at x. Then so are:

1. y = f(x) + g(x)
2. y = f(x) – g(x)
3. y = c f(x)
4. y = f(x) g(x)
5. y = $\frac{f(x)}{g(x)}$ provided g(x) ≠ 0
6. y = f(g(x)) provided f is continuous at g(x). (f need not otherwise be continuous at x)

Proof of 1

1. f(x) + g(x) exists, because by the continuity of f and g, f(x) and g(x) exist.
2. Let dx be any infinitesimal. Let y_1 = f(x) and y_2 = g(x) and y = $y_1 + y_2$. Then

   dy = (f(x+dx) + g(x+dx)) – (f(x) + g(x))

   = (f(x+dx)) – f((x)) + (g(x+dx) – g((x))

   = $dy_1 + dy_2$ since dy_1 = f(x+dx) – f(x), etc.

   = an infinitesimal.

End of Proof.

Proof of 4

1. f(x) g(x) exists because by the continuity of f and g, f(x) and g(x) exist.
2. Let dx be any infinitesimal. Then

   dy = f(x+dx) g(x+dx) – f(x) g(x)

   = (f(x)+dy_1) (g(x)+dy_2) – f(x) g(x) since dy_1 = f(x+dx) – f(x) ⇒ f(x+dx) = f(x) + dy_1, etc.

   = f(x) g(x) + f(x) dy_2 + g(x) dy_1 + $dy_1 dy_2$ – f(x) g(x)

   = f(x) dy_2 + g(x) dy_1 + $dy_1 dy_2$ = an infinitesimal

End of Proof.

Note. The proofs of the General Continuity Theorems using Cauchy's ϵ–δ definition are so difficult that some are often put into an appendix of textbooks or omitted entirely.

Proof of 6

1. f(g(x)) exists. Why?
2. Let dx be any infinitesimal. Then

   dy = f(g(x+dx)) – f(g(x))

   = f(g(x)+dy_2) – f(g(x))

   = f(g(x)) + dy_1^* – f(g(x)) by the continuity of f at g(x), dy_1^* is an infinitesimal.

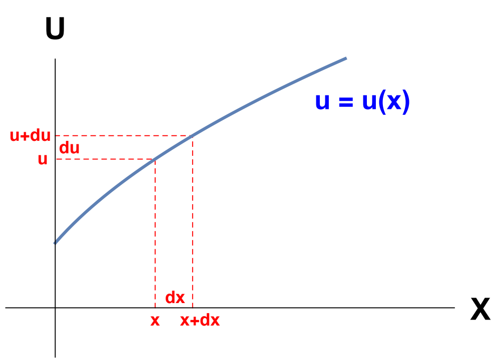
   = an infinitesimal.

End of Proof

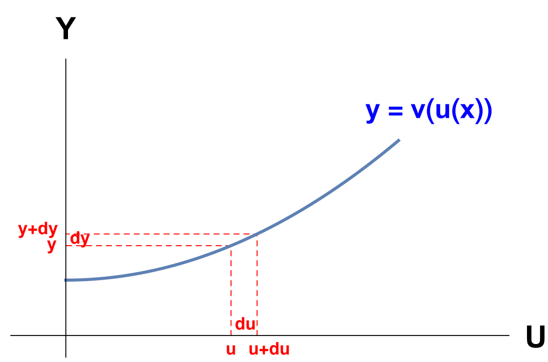
Graphical Demonstration of 6

1. By the continuity of v at x, v(x) exists.
   By the continuity of u at v(x), v(u(x)) exists.

2. Let dx be any infinitesimal, then by the continuity of u(x), du is an infinitesimal.
   Since du is an infinitesimal, by the continuity of v at u(x), dy is an infinitesimal.



u = u(x) makes the infinitesimal
dx into the infinitesimal du;
u is continuous at x.



y = v(u(x)) makes the infinitesimal
du into the infinitesimal dy;
y is continuous at u(x).

So for every infinitesimal dx, dy is an infinitesimal.

End of Demonstration

Continuity over an Interval

The function $y = \sqrt{x}$ we proved continuous for all $x > 0$. Since $\sqrt{x}$ is also continuous from the right at the domain endpoint $x = 0$, we agree to say that it is continuous on the interval $x \geq 0$. In general we say that a function is ***continuous over an interval*** if it is continuous at each point in the interval that is not an endpoint, and the appropriate one-sided continuity holds at any endpoints of the interval. (This is because we do not care about the continuity of a function where it does not exist.)

Using the Continuity Theorems

From the Basic Continuous Functions Theorems we know that the functions c and x are continuous for all x. Then by General Continuity Theorems part 3 so is 5x (taking $c = 5$) and consequently by part 1 so is $5x + 4$ (taking $c = 4$).

Likewise, since x is continuous for all x, by part 4 so is $x \cdot x = x^2$ and $x \cdot x^2 = x^3$ and, in general, x^n where n is a positive integer. Clearly:

Polynomial functions are continuous for all x

A **rational function** is one of the form $y = \frac{P(x)}{Q(x)}$ where P and Q are rational functions. Then by General Continuity Theorems part 5 we have the following, noting that points where $Q(x) = 0$ are not in the domain set.

Rational functions are continuous at each domain point

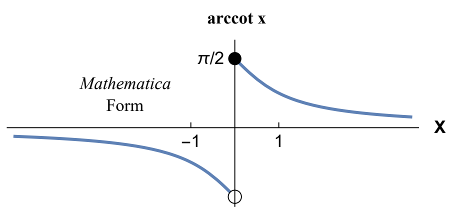
An **algebraic function** is one involving finite combinations of rational functions and roots. By Basic Continuous Theorems part 3 the continuity of $\sqrt[n]{x}$ is known. Again and using the convention about appropriate one–sided continuity at endpoints:

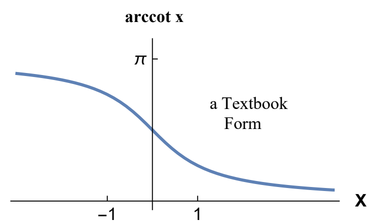
Algebraic functions are continuous at each domain point

Example The algebraic function $r(x) = 3x^2 - 4 + \sqrt{\frac{x}{x^2+4}}$ is continuous for $x \geq 0$.

The **elementary functions** are the basic continuous functions and finite combinations of them through addition, subtraction, multiplication, division, composition, algebraic inverses (e.g., log x and arcsin x) and piecewise defining except possibly at join points. The elementary functions are normally continuous at all domain points (except at join points of piecewise defined functions). However, inverse functions can be unpredictable because of how they are chosen (check out arccot x on your computer, calculator and other calculus textbooks).

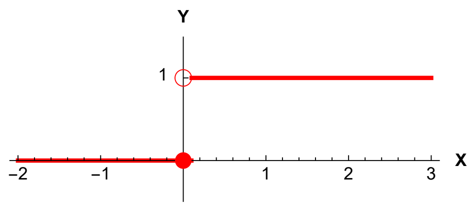
Example Below are two widely accepted versions of the function y = arccot x. The first is not continuous at x = 0; the second is. (You will have to wait for Calculus II to appreciate why this can happen.)





Example The elementary function $r(x) = \sin\left(3x^2 - 4 + \sqrt{\frac{x}{x^2+4}}\right)$ is continuous for $x \geq 0$.

Example The piecewise defined function $S(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$ is continuous except at the join point x = 0.



Exercises In exercises 1 to 7 prove the continuity of each stating the appropriate **Basic Continuous Functions Theorems** and **General Continuity Theorems** used.

#1. f(x) = 5 sin x

#2. f(x) = x + cos x

#3. f(x) = 2x + 5

#4. f(x) = 5 sin x (2x + 5)

#5. $f(x) = \frac{x + \cos x}{5 \sin x}$

#6. $f(x) = 7x^3 + \sqrt{\frac{x + \cos x}{5 \sin x}}$

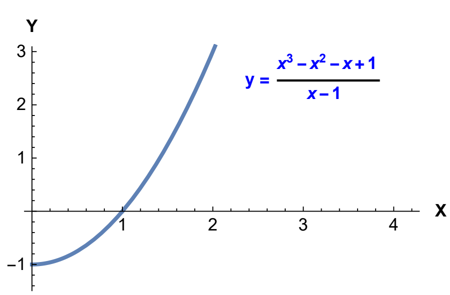
#7. Prove General Continuity Theorems, 2.

#8. Prove General Continuity Theorems, 3.

#9. Prove General Continuity Theorems, 5.

#10. In the proof of General Continuity Theorem 1, why does the existence of f(x) and g(x) imply the existence of f(x) + g(x)?

#11. Where is the function below not continuous?

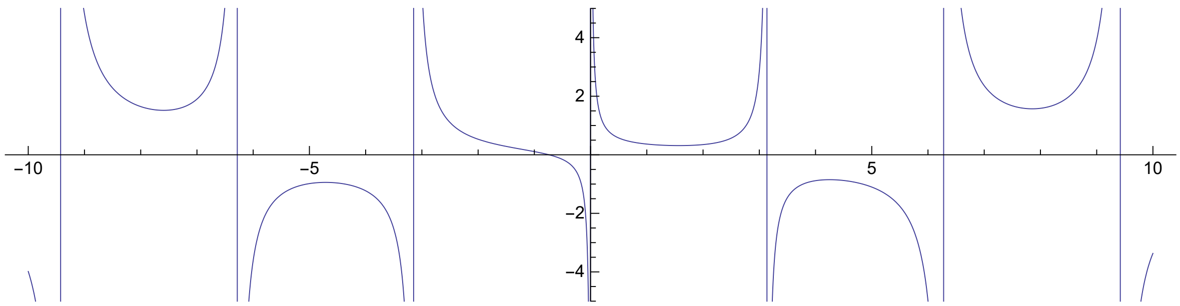


Solutions

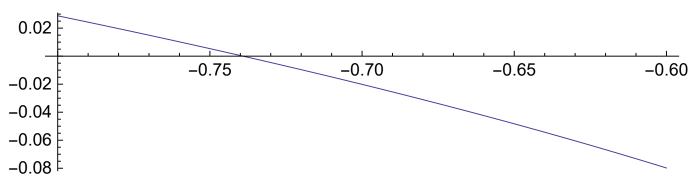
#1. By Basic Continuous Functions Theorems part 1, 5 is a continuous function and by part 4, sin x is a continuous function. Then by General Continuity Theorems part 4, so is their product 5 sin x.

#5. By exercises #1 and #2, 5 sin x and x + cos x are continuous functions. Then by General Continuity Theorems part 5, so is their quotient $\frac{x+\cos x}{5\sin x}$ at all domain points (x not a multiple of π).

#6. Hint: start by looking at part of the function in #5. Where is it not negative?



Graph near x = -1



#9. **Proof**

1. $\frac{f(x)}{g(x)}$ exists if $g(x) \neq 0$.
2. $dy = \frac{f(x+dx)}{g(x+dx)} - \frac{f(x)}{g(x)}$

$$= \frac{f(x)+dy_1}{g(x)+dy_2} - \frac{f(x)}{g(x)}$$

$$= \frac{\left(f(x)+dy_1\right) g(x) - f(x)\,(g(x)+dy_2)}{(g(x)+dy_2)\,g(x)}$$

$$= \frac{f(x)\,g(x)+g(x)\,dy_1 - f(x)\,g(x) - f(x)\,dy_2}{(g(x)+dy_2)\,g(x)}$$

$$= \frac{g(x)\,dy_1 - f(x)\,dy_2}{(g(x)+dy_2)\,g(x)}$$ Type $\frac{i}{h}$. $g(x) + dy_2 \neq 0$. Why?

= an infinitesimal.

End of Proof

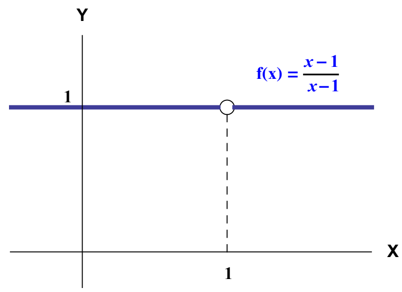
#10. By the Closure Property of the real numbers for addition.

1.3 The Theory of Limits. Limit Theorems

In this section we define *limit* in terms of continuity, develop an intuitive understanding of limits, and learn how to evaluate 'easy' limits. In the next section we will learn how to evaluate limits in general.

In computations, a calculus related function often does not exist at the point of interest but is otherwise well behaved near that point. Dealing with this problem involves what is called `finding the limit', the main new computation required to do calculus. Let us begin with two elementary examples that clearly illustrate the general problem.

Example 1 $f(x) = \frac{x-1}{x-1}$.



We first observe that the graph of f has a `hole' in it at $x = 1$. This is because $f(1) = \frac{1-1}{1-1} = \{\frac{0}{0}\}$,

which is *indeterminate* or *undefined*. It is called an ***indeterminate form*** because its value is not uniquely determined; consider the long division below. We get 7 with 0 remainder.

$$\begin{array}{r} 7 \\ 0\,\overline{)\,0} \\ \underline{0} \\ 0 \end{array}$$

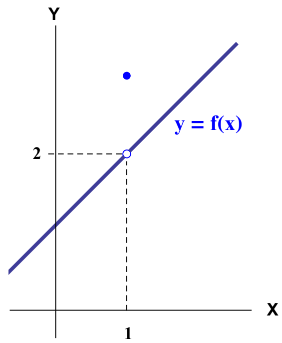
So $\{\frac{0}{0}\} = 7$. Need we say more! (We enclose indeterminate forms with braces to show they are not numbers.)

We cannot just cancel the $x - 1$ factors because that would give us $f(x) = 1$, technically a different function (because it has a different graph – no 'hole' in it). What we say to describe the situation that while $f(1)$ is not defined, infinitesimally near $x = 1$ $f(x)$ is infinitesimally close to 1, is 'the limit as x approaches 1 of $\frac{x-1}{x-1}$ is 1' and write

$$\lim_{x \to 1} \frac{x-1}{x-1} = 1.$$

This example illustrates the unique difficulty that occurs in beginning calculus, the problem of finding the derivative of a function, which you will encounter in a few more lessons. **The limit is essentially the value that fills in the 'hole' in the graph to make the resulting function continuous there.**

Example 2 $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$. Its graph is shown below.



Clearly, from the graph of f, we expect $\lim_{x\to 1} f(x) = 2$, the value that would fill in the 'hole'. If we were to redefine the function so that $f(1) = 2$, then the function would be continuous. We do not encounter this type of difficulty frequently in the calculus. But again we write

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = 2.$$

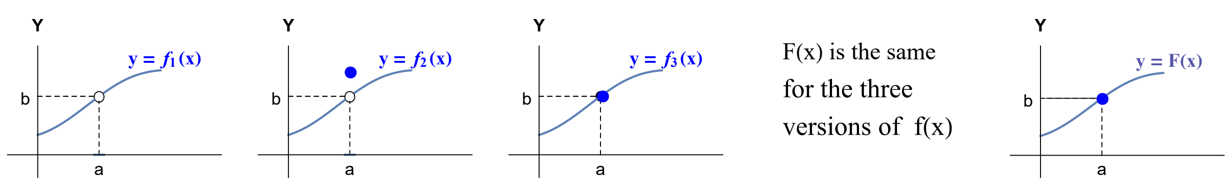
When we state the formal definition of limit, we will want to cover both types of problems. Intuitively $\lim_{x\to a} f(x)$, if it exists, is the rounded off value of $y = f(x)$ infinitesimally close to $x = a$.

In both of the previous examples we were able to define or redefine the function at $x = 1$ so that the 'hole' is filled in, that is, so that the curve is continuous. This suggests the following precise definition of limit.

Continuity Definition of Limit The ***limit as x approaches a of f(x) equals b***, written $\lim_{x\to a} f(x) = b$ means the function

$$F(x) = \begin{cases} f(x), & x \neq a \\ b, & x = a \end{cases}$$

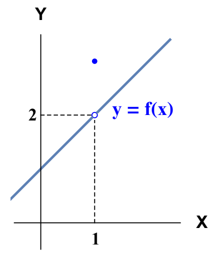
is continuous at $x = a$.



Note: this definition does not provide general method of finding the limit b although you can sometimes guess it by looking at the graph and answering the question, "When x is infinitesimally close to a, what real number b is y infinitesimally close to or what value of F(a) would make F(x) continuous at x = a? In the next section, we will give a hyperreal definition of limit which is more useful in determining b in all circumstances.

This definition allows us to immediately translate continuity theorems into limit theorems.

Example 3 Let us find the limit of the previous example using this definition. $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$.



Proof From its graph it looks like $\lim_{x\to 1} f(x) = 2$. To prove this, consider

$$F(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

$$= \begin{cases} \frac{(x-1)(x+1)}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases} \qquad \text{factoring}$$

$$= \begin{cases} x+1, & x \neq 1 \\ 2, & x = 1 \end{cases} \qquad \text{can cancel since } x \neq 1$$

$$= x+1 \qquad \text{in both cases since } x+1 = 1+1 = 2 \text{ when } x = 1$$

which from the previous section we know to be a continuous function at $x = 1$. **End of Proof**

Note about terminology The notation $\lim_{x\to a} f(x) = b$ read 'the limit as x approaches a of f(x) is b' suggests one finds b by checking the values of f(x) as x gets closer and closer to a. An organized way of doing this is computing the value of f(x) for a sequence of real number x-values approaching a: $x_1, x_2, x_3, x_4, \cdots \to a$. Then $f(x_1), f(x_2), f(x_3), f(x_4), \cdots \longrightarrow b$. The limit exists only if the same result b is obtained for every for every such sequence.

As a hyperreal literate person, you might wonder why anyone would spend lots of time piddling around with real numbers when b, if it exists, is just the rounded off value of $f(a+i_o)$, say. You are right of course. However, hyperreal numerical computations are often difficult to do because hyperreal calculators are not readily available. Also, since limit notation is universally used, we will too. It will turn out that using sequences of real numbers approaching $x = a$ can actually be a practical way of finding the approximate value of an otherwise intractable limit.

Limit Theorems

There are three basic theorems regularly used for the efficient evaluation of limits. The first says that the limit of a continuous function is always easy to find. The next two follow directly from this theorem and the continuity theorems of the previous section.

Limit of a Continuous Function Theorem (Easy Limits Theorem) Suppose f is continuous at $x = a$. Then

$$\lim_{x\to a} f(x) = f(a)$$

Proof

$$F(x) = \begin{cases} f(x), & x \neq a \\ f(a), & x = a \end{cases}$$

$= f(x)$, which is continuous at $x = a$. **End of Proof**

This theorem says if a function is continuous, then finding limits is easy; you just 'plug' the value of a into f(x).

Example

$$\lim_{x\to 2}\frac{x^3+3x+2}{2x-1} = \frac{2^3+3\cdot 2+2}{2\cdot 2-1} = \frac{16}{3}$$

Basic Limit Theorems

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, n a positive integer; $a > 0$ if n even.
4. $\lim_{x\to a} \sin x = \sin a$
5. $\lim_{x\to a} \cos x = \cos a$

Proofs These all follow from the 'Easy Limit Theorem.' Mentally verify this.

General Limit Theorems Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

Then 1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
3. $\lim_{x\to a}(c\,f(x)) = c\cdot\lim_{x\to a} f(x)$.
4. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$.
5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.
6. $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$, provided f is continuous at g(a). $\lim_{x\to a} f(x)$ need not **exist.**

Proofs These all follow from the 'Easy Limit Theorem'. Mentally verify this.

Example 4

$\lim_{x\to 5}\frac{x^3+4}{2x+3}$ **continuous at $x = 5$**

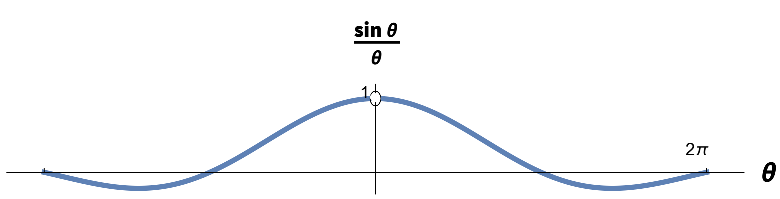
$= \frac{5^3+4}{2\cdot 5+3}$ **Limit of a Continuous Function Theorem**

$= \frac{129}{13}$

Two Important Trig Limits

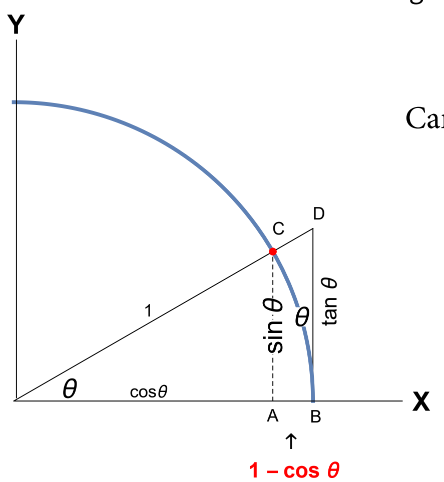
$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Let us graph the function $\frac{\sin\theta}{\theta}$ is shown below. It looks like the limit is 1. This limit is so important it deserves a a careful examination.



The usual proof in calculus textbooks is a geometric one using the formula for the area of a circle: $A = \pi r^2$. Unfortunately you were told that formula early in school without proof. You will derive the formula in second semester calculus. The 'usual proof' is an exercise.

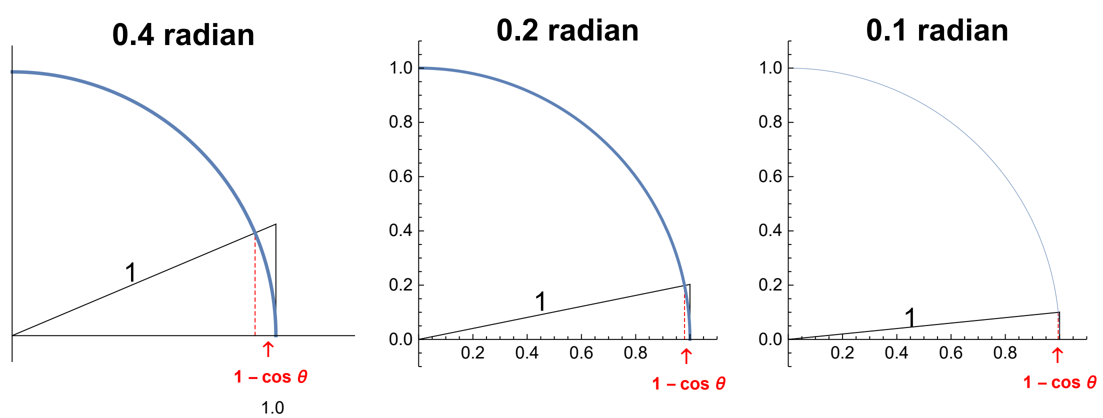
We will do a detailed geometric look at what happens to the ratio $\frac{\sin\theta}{\theta}$ as $\theta \to 0$.
Let us go back to the unit circle with the main three trig functions identified.



Can you see what happens to the arc BC as $\theta \to 0$?

Answer: it approaches the $\tan\theta$ line.

Next, another look, 3 angles approaching 0.



It looks like as $\theta \to 0$, both the $\sin\theta$ line and the arc θ approach the $\tan\theta$ line.
Look at the above graph with $\theta = 0.4$, 0.2 and 0.1 radians. This type of understanding is a valuable tool as you go through future calculus.

Example

$\lim_{x\to 0} \frac{\sin(2x)}{x}$ want to use above limit

$= 2 \cdot \lim_{2x\to 0} \frac{\sin(2x)}{2x}$ doctoring up

$= 2 \lim_{\theta\to 0} \frac{\sin\theta}{\theta}$ letting $\theta = 2x$

$= 2 \cdot 1$

$= 2$

Another important trig limit is

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta}$$

Proof

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta}$$

$$= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta}$$

$$= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta\,(1 + \cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta\,(1 + \cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1 + \cos\theta}$$

$$= 1 \cdot \frac{0}{1 + 1} = 0$$

End of Proof

Intuitive Summary Recall that

$\lim_{x \to a} f(x) = b$

means that the values of f(x) 'just to the left' and 'just to the right' of $x = a$ are infinitesimally close to the common value b. A better notation than $\lim_{x \to a} f(x) = b$ might be NV(f(a)) = b, ***the neighboring value of f at x = a is b***; but it won't catch on.

If you know that the limit exists, you need check only one value of x infinitesimally close to a. If two different values of x infinitesimally close to a round off to different numbers, then $\lim_{x \to a} f(x)$ does not exists.

Exercises

In exercises 1 to 3, use the definition of limit to prove the limit statement. Model like Example 3.

#1. Prove $\lim_{x\to 3}\frac{2x-6}{x-3} = 2$

#2. Prove $\lim_{x\to 4}\frac{x^2-7x+12}{x-4} = 1$

#3. Prove $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-x} = -1$

In the following use the Basic Limits Theorem to evaluate each quickly.

#4. $\lim_{x\to 5} 17 =$

#5. $\lim_{x\to \pi}\sin x =$

#6. $\lim_{x\to 1} (2x+1)\cos x =$

In numbers 7 to 9 use the Limit of a Continuous Function Theorem.

#7. $\lim_{x\to 0} (x+1)^2\cos(2x) =$

#8. $\lim_{x\to 0}\sqrt{(x+1)^2\cos(2x)+5}\ \sin x =$

#9. $\lim_{x\to 4}\frac{x^2-7x-12}{x-3} =$

Note there is no practical difference between using the two Limit Theorems and and the Limit of a Continuous Function Theorem.

#10. Verify that the Basic Limits Theorem reflects special cases of the Limit of a Continuous Function Theorem.

In the following evaluate each limit using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, trig identities, and the Continuity Theorems.

#11. a. $\lim_{x\to 0}\frac{\sin(5x)}{2x}$ b. $\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}$

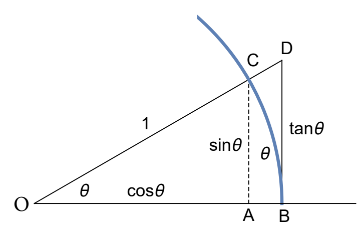
#12. a. $\lim_{x\to 0}\frac{\tan x}{x}$ b. $\lim_{x\to 0}\frac{x}{\sin x+\tan x}$

#13. a. $\lim_{x\to 0} x\csc x$ b. $\lim_{x\to 0}\tan x$

#14. a. $\lim_{\theta\to 0}\frac{1-\cos(2\theta)}{\theta}$ b. $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$

#15. Prove geometrically that $\lim_{h\to 0}\frac{\sin h}{h}$

Assume the area of a sector formula: $A = \frac{1}{2}r^2\theta$. Consider the unit circle.



Observe:

Area triangle OAC $\leq$ area sector OBC $\leq$ area triangle OBD.

Using this as the starting point, derive the limit formula.

#16. a. Prove $\lim_{x\to 2}\frac{x^2-3x+2}{x-2} \neq 5$ by showing the corresponding F(x) is not continuous at x = 2.

b. Discover a way of choosing F(2) so that F is continuous at x = 2.

Solutions

Use the definition of limit to prove each limit statement. Model your solutions after example 3.

#1. Prove $\lim_{x\to 3}\frac{2x-6}{x-3} = 2$

Proof Consider

$$F(x) = \begin{cases} \frac{2x-6}{x-1}, & x \neq 3 \\ 2, & x = 3 \end{cases}$$

$$= \begin{cases} \frac{2(x-3)}{x-3}, & x \neq 3 \\ 2, & x = 3 \end{cases}$$ factoring

$$= \begin{cases} 2, & x \neq 3 \\ 2, & x = 3 \end{cases}$$ can cancel since $x \neq 1$

$$= 2$$

which we know to be a continuous function at $x = 3$. **End of Proof**

#3. Prove $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-x} = -1$

Proof Consider

$$F(x) = \begin{cases} \frac{x^2-3x+2}{x^2-x}, & x \neq 1 \\ -1, & x = 1 \end{cases}$$

$$= \begin{cases} \frac{(x-1)(x-2)}{x(x-1)}, & x \neq 1 \\ -1, & x = 1 \end{cases}$$ factoring

$$= \begin{cases} \frac{x-2}{x}, & x \neq 1 \\ -1, & x = 1 \end{cases}$$ can cancel since $x \neq 1$

$$= \frac{x-2}{x}$$ since $\frac{x-2}{x} = -1$ when $x = 1$

which is a continuous function at $x = 1$. **End of Proof**

#5. $\lim_{x\to\pi}\sin x = \sin\pi = 0$ (Easy Limit)

#7. $\lim_{x\to 0}(x+1)^2\cos(2x) = (0+1)^4\cos(2\cdot 0) = 1$ (Easy Limit)

#9. $\lim_{x\to 4}\frac{x^2-7x-12}{x-3} = \frac{4^2-7\cdot 4+2}{4-3} = -24$ (Easy Limit)

#11. a. $\lim_{x\to 0}\frac{\sin(5x)}{2x} = \frac{5}{2}\lim_{5x\to 0}\frac{\sin(5x)}{5x} = \frac{5}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{5}{2}\cdot 1 = \frac{5}{2}$

#13. a. $\lim_{x\to 0} x\csc x = \lim_{x\to 0}\frac{x}{\sin x} = \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = 1$

1.4 The Practical Computation of Limits

Finding Limits

The main problem in the previous section with evaluating a difficult limit $\lim_{x \to a} f(x,)$ is that somehow you must guess the limit b in advance and then use the definition of limit prove that your guess was correct! Normally the main problem in calculus will be finding the limit number b, not in proving that $\lim_{x \to a} f(x) = b$ by showing that $F(x) = \begin{cases} f(x), & x \neq a \\ b, & x = a \end{cases}$ is a continuous function at $x = a$.

There are three common ways of evaluating difficult limits, at least one of which will evaluate any given limit:

1. **Analytically** This is the precise method and therefore preferred. However, often this is not possible.

   Hyperreal Style Use the hyperreal (computational) definition of limit below. In theory this method always works; in practice it often does not.

   Limit Style. Reduce, by algebra the function to one which is continuous at $x = a$, an easy problem. It is about as limited as the hyperreal method.

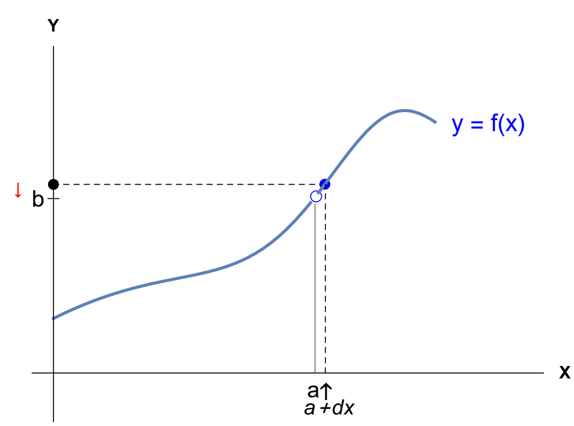
2. **Geometrically** This method is quick and intuitive if you have a CAS or perhaps a graphing calculator. Just examine the graph very close to $x = a$. It may only give an approximation to the limit. Zooming in is often required.

3. **Numerically** Examine $f(x)$ for a sequence of real values approaching $x = a$. This method is tedious and risky; another sequence might give a different result in some difficult problems, which means the limit does not exist.

The last two methods normally give only an approximation to the limit, however, to as many decimal places as you wish. The advantage of these last two methods is they work for almost any function you will encounter.

Recall that the limit, if it exists, is the common rounded off value of $f(x)$ infinitesimally close to $x = a$. This suggests an alternate definition of limit which is helpful in finding the limit b. All the methods of finding limits above one way or the other are versions of this alternate definition.

Hyperreal definition of limit $\lim_{x \to a} f(x) = b$ means $f(a+dx) \approx> b$ for every infinitesimal $dx \neq 0$.



Limits Analytically - Hyperreal Style

Example 1 $\lim_{x\to 5} \frac{7x^2-11x-120}{x^2-x-20}$ $\left\{\frac{0}{0}\right\}$

Let dx be any nonzero infinitesimal. Then

$\frac{7(5+dx)^2-11(5+dx)-120}{(5+dx)^2-(5+dx)-20}$

$= \frac{59\,dx+7\,dx^2}{9\,dx+dx^2}$ expanding and simplifying

$= \frac{dx(59+7\,dx)}{dx(9+dx)}$ factoring

$= \frac{59+7\,dx}{9+dx}$ can cancel since $dx \neq 0$

$\approx> \frac{59}{9}$.

Limits Analytically - Limit Style

Continuity Definition of Limit The ***limit*** as x approaches a of f(x) equals b, written

$$\lim_{x\to a} f(x) = b$$

means the function

$$F(x) = \begin{cases} f(x), & x \neq a \\ b, & x = a \end{cases}$$

is continuous at $x = a$.

We adapt the method of the continuity definition and ignore the F notation by agreeing never to allow x to be equal to a; then if we can convert f(x) algebraically to a function which is a continuous function; we can then evaluate the limit by setting $x = a$.

Now, let us illustrate this Example 1 using this concise limit style.

$\lim_{x\to 5} \frac{7x^2-11x-120}{x^2-x-20}$

$= \lim_{x\to 5} \frac{(x-5)(7x+24)}{(x-5)(x+4)}$ a non-trivial factorization

$= \lim_{x\to 5} \frac{7x+24}{x+4}$ can cancel since $x \neq 5$

$= \frac{7\cdot 5+24}{5+4}$ limit of a continuous function

$= \frac{59}{9}$.

The limit style method is to reduce the expression, by canceling equal factors which approach 0, to one for which the **Limit of a Continuous Function (Easy Limits)** theorem applies. Most mathematicians use this method even though the equivalent hyperreal method is sometimes easier because less factoring skill is required. Nevertheless we will often use the limit style because most mathematicians do.

Still, we will always understand that the limit is essentially the value of f infinitesimally close to $x = a$; the limit idea of `approaching' the answer by looking at real number approximations as `x gets closer and closer' should seem quite a second rate idea.

Example 2 Limit of a rational indeterminate form

$\lim_{x\to 1}\frac{x^2-1}{x-1}$ indeterminate form $\left\{\frac{0}{0}\right\}$

Hyperreal Method

$\frac{(1+dx)^2-1}{(1+dx)-1}$

$= \frac{1+2\,dx+dx^2-1}{dx}$

$= \frac{dx(2+dx)}{dx}$

$= 2+dx$ $dx \neq 0$

$\approx> 2.$

Traditional Limit Style

$\lim_{x\to 1}\frac{x^2-1}{x-1}$

$= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$ factoring

$= \lim_{x\to 1}(x+1)$ can cancel because $x \neq 1$ in the limit process

$= 1+1$ **Limit of a Continuous Function**

$= 2$

Note that we removed the cause of the $\left\{\frac{0}{0}\right\}$ indeterminate form when we canceled the $x-1$ factors. The beauty of these two methods is that not only does it find the limit, but a proof using the definition of limit is unnecessary; we are sure it is correct because we used correct hyperreal algebra. We note that the limit method works only if we can algebraically reduce the difficult limit to the easy continuous case.

Example 3 Limit of an algebraic indeterminate form

$\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}$ indeterminate form $\left\{\frac{0}{0}\right\}$

$= \lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$ rationalizing the numerator

$= \lim_{x\to 0}\frac{(x+9)-9}{x\left(\sqrt{x+9}+3\right)}$

$= \lim_{x\to 0}\frac{x}{x\left(\sqrt{x+9}+3\right)}$

$= \lim_{x\to 0}\frac{1}{\sqrt{x+9}+3}$ can cancel because $x \neq 0$

$= \frac{1}{\sqrt{9}+3}$ **Limit of a Continuous Function (Easy Limit)**

$= \frac{1}{6}$

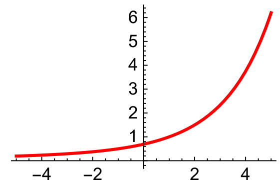
Example 4 A limit that cannot be found exactly (at this time)

$\lim_{x\to 0}\frac{2^x-1}{x}$ $\left\{\frac{0}{0}\right\}$

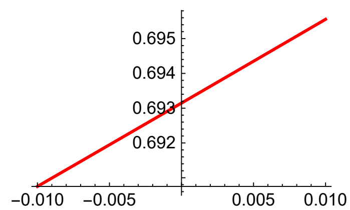
We have no way of factoring out the troublesome x in the numerator. So we cannot cancel the x's and evaluate a resulting easy limit.

Limits Graphically Below are examples of functions whose limits are difficult or impossible to evaluate analytically because 'factoring and canceling' is not possible. We examine the graph 'close' to the limit point x = a.

Example 5 $\lim_{x\to 0} \frac{2^x - 1}{x}$ indeterminate form $\left\{\frac{0}{0}\right\}$



Note: Because of the way the computer graphs (by connecting points with line segments), it looks like the function is continuous at $x = 0$ even though there must be a 'hole' in the graph at $x = 0$. It looks like $\lim_{x\to 0} \frac{2^x - 1}{x} \doteq 0.7$. To get a better approximation, zoom in.



$\lim_{x\to 0} \frac{2^x - 1}{x} \doteq 0.693.$

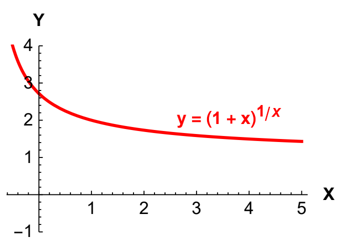
This answer is approximate. The exact answer you will learn in the next calculus course is $\log_e 2 \doteq 0.693\cdots$, which one cannot determine graphically.

Example 6 Another indeterminate form

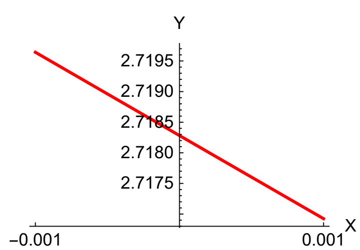
$\lim_{x\to 0}(1 + x)^{1/x}$ indeterminate form $\{1^\infty\}$

We will discuss the use of the limit symbol ∞ in the next lesson. For now, take it to mean extremely large.

First, why is $\{1^\infty\}$ indeterminate? 1 to any power is 1; $1^{1000} = 1$. But a number slightly larger than 1 raised to a very large number can be a large number; $1.01^{1000} \doteq 20{,}959.2$. Likewise a number slightly smaller than 1 raised to a very large number can be a number close to 0; $0.99^{1000} \doteq 0.0000431712$.



It looks like $\lim_{x\to 0}(1 + x)^{1/x} \doteq 3$. Let's zoom in near $x = 0$.



$\lim_{x\to 0}(1 + x)^{1/x} \doteq 2.718.$

Note that this answer again is approximate. The exact limit is required in the definition of limit; however, for many applications 2.718 may be good enough.

Limits Numerically Because you do not have visual cues about the behavior on either side of x = a, you should be especially careful with this method. This is a crude way of trying to examine the value of f(x) infinitesimally close to the limit value x = a. The hyperreal definition of limit suggests it is efficient to choose a sequence that approaches **a** rapidly. For example if a = 0, the sequence

$$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$$

is more efficient than

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Example 7 $\lim_{x\to 0} \sin \frac{\pi}{x}$

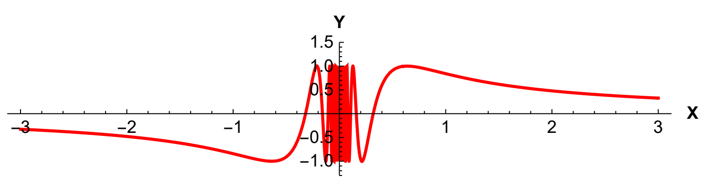
Let us examine this function for two *carefully selected* sequences approaching 0.

x	$\sin \frac{\pi}{x}$
1/2	0
1/4	0
1/8	0
↓	↓
0	?
↑	↑
- 1/8	0
- 1/4	0
- 1/2	0

From the sequences above, one is tempted to deduce that $\lim_{x\to 0} \sin \frac{\pi}{x} = 0$. But from the graph below, clearly

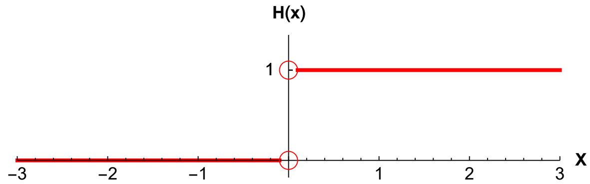
$\lim_{x\to 0} \sin \frac{\pi}{x}$ does not exist.

Be sure that your sequences are not `carefully' selected! (This error cannot happen, of course, if the limit exists.)



To be 100% certain of getting the correct answer, you should check values of the function for *every* sequence approaching a. Practically, using one sequence approaching a from the left and one approaching a from the right should normally be sufficient. In a 'tricky' limit, you may wish to graph the function to see if extra care is required and, of course, do not use carefully selected sequences.

Example 8 $\lim_{x\to 0} H(x)$



Using a sequence approaching 0 from the left it looks like $\lim_{x\to 0} H(x) = 0$.
Using a sequence approaching 0 from the right it looks like $\lim_{x\to 0} H(x) = 1$.
So $\lim_{x\to 0} H(x) = 0$ does not exist.

Example 9

$\lim_{x\to 0} \frac{4^x - 3^x}{x}$ indeterminate form $\left\{\frac{0}{0}\right\}$

First we construct a table of values with $x \to 0$ from both sides.

x	$\frac{4^x - 3^x}{x}$
1	1.00000
.1	.32575
.01	.29128
.001	.28804
.0001	.28772
↓	↓
0	?
↑	↑
−.0001	.28765
−.1	.25408
−1	.08333

Clearly $\lim_{x\to 0} \frac{4^x - 3^x}{x} \doteq 0.2876$ (trusting that the function does not have strange behavior between table entries).
If you suspect the limit exists and will settle for a rough approximation of the limit, it is not unreasonable to try just one value `near' 0 numerically:

$$\frac{4^{0.0001} - 3^{0.0001}}{0.0001} \doteq 0.288$$

Note: Generally the graphical method is quicker, more intuitive and less error prone than the numerical sequence method.

Exercises

In the following, identify the indeterminate form and find the limit algebraically.

#1. a. $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$ b. $\lim_{x\to 3}\frac{x^2-3x}{x^2-x-6}$

#2. a. $\lim_{x\to 1}\frac{x^3-1}{x^5-1}$ b. $\lim_{x\to 2}\frac{x^3-8}{x^2-4}$

#3. a. $\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}$ b. $\lim_{x\to 0}\frac{x}{\sqrt{4-2x}-2}$

#4. a. $\lim_{h\to 0}\frac{(2+h)^2-4}{h}$ b. $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

In the following, identify the indeterminate form and find the limit graphically. Observe that these may be difficult to do algebraically.

#5. $\lim_{x\to 0}\left(1+\frac{x}{2}\right)^{1/x}$

#6. a. $\lim_{x\to 0}\frac{\sqrt{x^2}-x}{x}$

In the following, identify the indeterminate form and find the limit numerically.

#7. $\lim_{x\to 0}\frac{3^x-1}{x}$

#8. $\lim_{x\to 1}\frac{\sin(\pi x)}{x-1}$

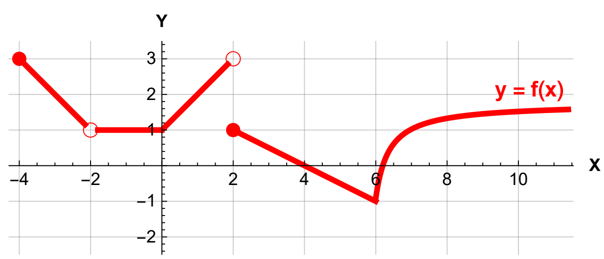
In the following, identify the indeterminate form and use any method to evaluate.

#9. a. $\lim_{x\to 4}\frac{2x-4}{(x^2+16)^3}$ b. $\lim_{t\to 3}\frac{4+\sqrt{t}}{t}$

#10. a. $\lim_{t\to 4}\frac{t-4}{\sqrt{t}-2}$ b. $\lim_{x\to 2}\frac{1}{x-2}\left(\frac{1}{x}-\frac{1}{2}\right)$

#11. Work Example 3 by the hyperreal definition.

#12. f(x) is the function shown below.



$\lim_{x\to -3}f(x) =$ $\lim_{x\to -3.5}f(x) =$ $\lim_{x\to 0}f(x) =$ $\lim_{x\to 4}f(x) =$ $\lim_{x\to 6}f(x) =$

$\lim_{x\to 7}f(x) =$ $\lim_{x\to -2}f(x) =$ $\lim_{x\to 5}f(x) =$ $\lim_{x\to 2}f(x) =$ $\lim_{x\to 20}f(x) =$

#13. Evaluate Example 1 taking $dx = i_o = 0.000\cdots001{,}000\cdots$.

Solutions

#1. a. $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$ $\left\{\frac{0}{0}\right\}$

$= \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2}$

$= \lim_{x\to 2} x+3$

$= 2+3$

$= 5$

#2. a. $\lim_{x\to 1} \frac{x^3-1}{x^5-1}$ $\left\{\frac{0}{0}\right\}$

$= \lim_{x\to 1} \frac{(x-1)\left(x^2+x+1\right)}{(x-1)\left(x^4+x^3+x^2+x+1\right)}$

$= \lim_{x\to 1} \frac{x^2+x+1}{x^4+x^3+x^2+x+1}$

$= \frac{1+1+1}{1+1+1+1+1}$

$= \frac{3}{5}$

#3. b. $\lim_{x\to 0} \frac{x}{\sqrt{4-2x}-2}$ $\left\{\frac{0}{0}\right\}$

$= \lim_{x\to 0} \frac{x}{\sqrt{4-2x}-2} \frac{\sqrt{4-2x}+2}{\sqrt{4-2x}+2}$

$= \lim_{x\to 0} \frac{x\left(\sqrt{4-2x}+2\right)}{(4-2x)-4}$

$= \lim_{x\to 0} \frac{\sqrt{4-2x}+2}{-2}$

$= \frac{2+2}{-2}$

$= -2$

#4. a. $\lim_{h\to 0} \frac{(2+h)^2-4}{h}$ $\left\{\frac{0}{0}\right\}$

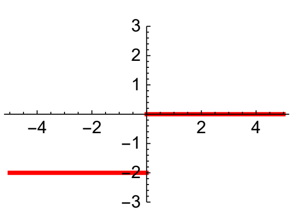
$= \lim_{h\to 0} \frac{4+4h+h^2-4}{h}$

$= \lim_{h\to 0} \frac{h(4+h)}{h}$

$= \lim_{h\to 0} 4+h$

$= 4$

#6.



$\lim_{x\to 0} \frac{\sqrt{x^2}-x}{x}$ does not exist .

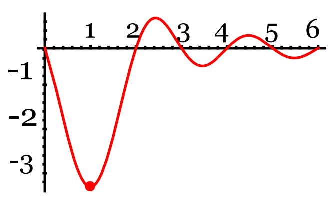
#8.

$\lim_{x\to 1}\frac{\sin(\pi x)}{x-1}$ $\left\{\frac{0}{0}\right\}$

x	$\frac{\sin(\pi x)}{x-1}$
0.9	– 3.09017
0.99	– 3.14108
0.999	– 3.14159
↓	↓
1	1
↑	↑
1.001	– 3.14159
1.01	– 3.14108
1.1	– 3.09017

It looks like $\lim_{x\to 1}\frac{\sin(\pi x)}{x-1} = -\pi$.

Graphical check:



#9. b. $\lim_{t\to 3}\frac{4+\sqrt{t}}{t}$ not indeterminate (continuous at $x = 3$)

$= \frac{4+\sqrt{3}}{3}$

#10. a. $\lim_{t\to 4}\frac{t-4}{\sqrt{t}-2}$

$= \lim_{t\to 4}\frac{(\sqrt{t}-2)(\sqrt{t}+2)}{\sqrt{t}-2}$ think of $t-4$ as the difference of squares

$= \lim_{t\to 4}(\sqrt{t}+2)$

$= \sqrt{4}+2$

$= 4$

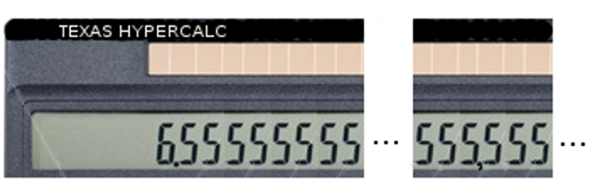
#12. 2 2.5 1 0 −1

≐ 1 1 −1/2 DNE 1.9?

#13. Evaluating $\frac{7x^2-11x-120}{x^2-x-20}$ at $x = 5 + i_o$ = 5.000 · · · 001,000 · · · by long division we get

6.555 · · · 555,555 · · · ~> 6.555 · · · = $\frac{59}{9}$

With the the aid of a hyper-calculator, or www.lightandmatter.com/calc/inf we obtain the same result.



1.5 Extensions of the Limit Idea: One–sided Limits. Infinite Limits

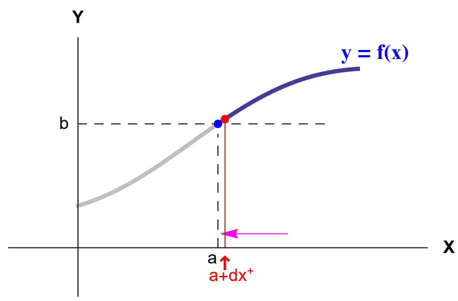
We begin with a full discussion of one-sided limits. Also what happens if on one or both sides the function attains arbitrarily large values.

One–sided Limits

Hyperreal Definition The ***limit from the right*** of f(x) at $x = a$

$\lim_{x \to a^+} f(x) = b$

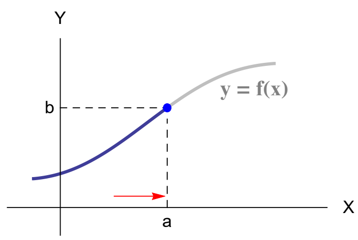
means $f(a + dx^+) \approx b$ for every positive infinitesimal dx^+.



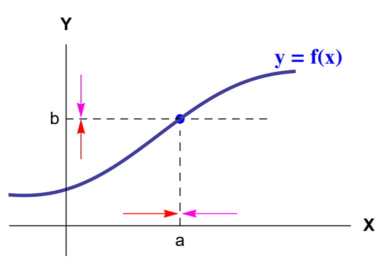
Hyperreal Definition The ***limit from the left*** of f(x) at $x = a$

$\lim_{x \to a^-} f(x) = b$

means $f(a - dx^+) \approx b$ for every positive infinitesimal dx^+.



Two–sided Limits Theorem $\lim_{x \to a} f(x) = b$ means $\lim_{x \to a^-} f(x) = b$ and $\lim_{x \to a^+} f(x) = b$.

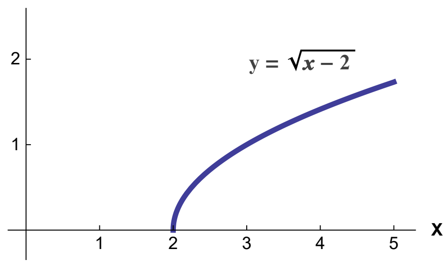


This theorem says that if the limit exists at $x = a$, the function approaches the same value $y = b$ from either side and conversely.

We often evaluate easy one and two-sided limits (those which do not lead to an indeterminate form) quickly by examining the graph of the function whose limit is being taken near $x = a$.

End-Point Agreement We will adopt the convention that a limit exists at the endpoint of the domain of a function if the appropriate one-sided limit exists. Some mathematicians do not observe this agreement. We may in some circumstances, with proper warning, do the same.

Example



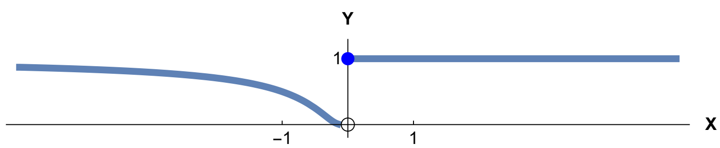
From the graph:

$\lim_{x \to 2^-} \sqrt{x-2}$ does not exist

$\lim_{x \to 2^+} \sqrt{x-2} = 0$

$\Longrightarrow \lim_{x \to 2} \sqrt{x-2} = 0$ (endpoint agreement)

Example $f(x) = \begin{cases} 2^{1/x}, & x < 0 \\ 1, & x \geq 0 \end{cases}$



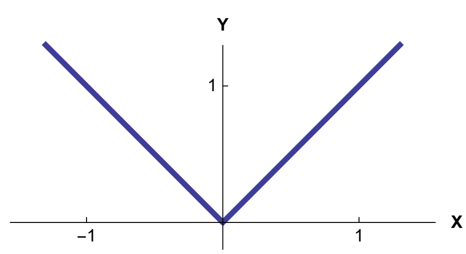
From the graph:

$\lim_{x \to 0^-} f(x) = 0$

$\lim_{x \to 0^+} f(x) = 1$

$\Longrightarrow \lim_{x \to 0} f(x)$ does not exist

Example $f(x) = |x|$



From the graph:

$\lim_{x \to 0^-} |x| = 0$

$\lim_{x \to 0^+} |x| = 0$

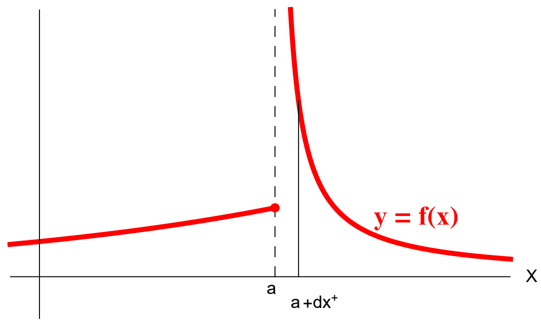
$\Longrightarrow \lim_{x \to 0} |x| = 0$

Infinite Limits Sometimes a function becomes unbounded near $x = a$. We use the symbols $\pm\infty$ to express certain types of unboundedness.

Hyperreal Definition The limit as x approaches a from the right is ***plus infinity***, written

$$\lim_{x \to a^+} f(x) = +\infty,$$

means $f(a + dx^+) \approx> +\infty$ for every positive infinitesimal dx^+.

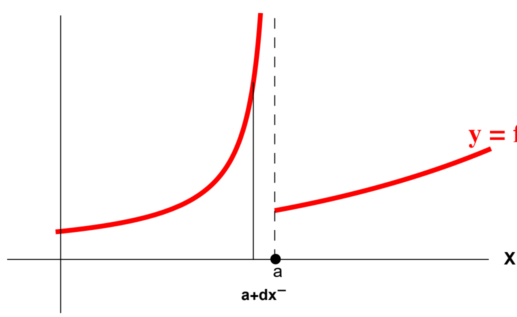


Hyperreal Definition The limit as x approaches a from the left is ***plus infinity***, written

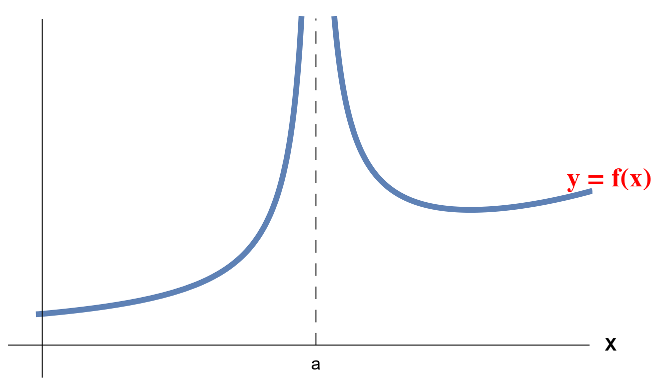
$$\lim_{x \to a^-} f(x) = +\infty,$$

means

$$f(a - dx^+) \approx> +\infty \text{ for every positive infinitesimal } dx^+.$$

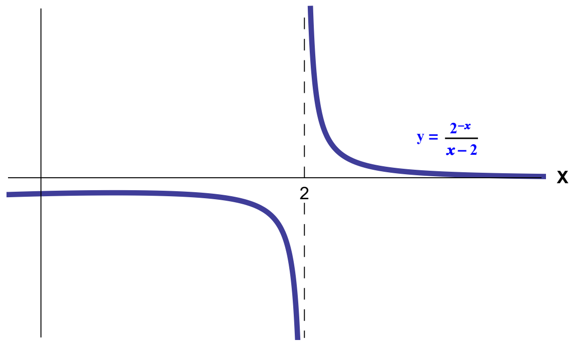


Definition $\lim_{x \to a} f(x) = +\infty$ means $\lim_{x \to a^-} f(x) = +\infty$ and $\lim_{x \to a^+} f(x) = +\infty$.



We make similar definitions for $\lim_{x \to a^-} f(x) = -\infty$, $\lim_{x \to a^+} f(x) = -\infty$ and $\lim_{x \to a} f(x) = -\infty$.

Definition If $\lim_{x\to a^-} f(x) = \pm\infty$ *or* $\lim_{x\to a^+} f(x) = \pm\infty$, we say that the line $x = a$ is a *vertical asymptote* to the curve $y = f(x)$.



From the graph:

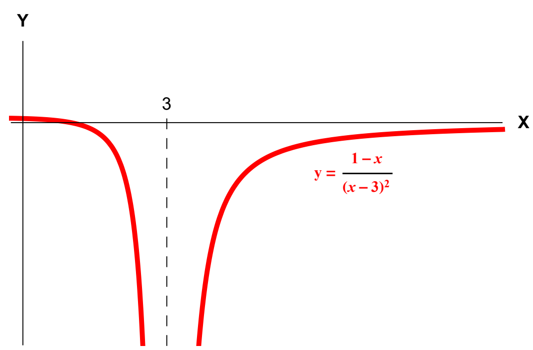
$\lim_{x\to 2^-} \frac{2-x}{x-2} = -\infty$

$\lim_{x\to 2^+} \frac{2-x}{x-2} = +\infty$

$\Longrightarrow \lim_{x\to 2} \frac{2-x}{x-2}$ does not exist because the one–sided limits are different

However, $x = 2$ is a vertical asymptote to $y = \frac{2-x}{x-2}$.

Example



From the graph:

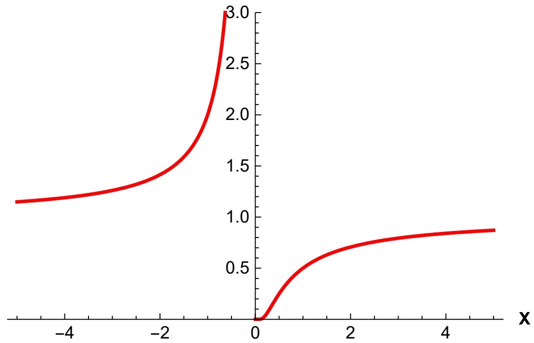
$\lim_{x\to 3^-} \frac{1-x}{(x-3)^2} = -\infty$

$\lim_{x\to 3^+} \frac{1-x}{(x-3)^2} = -\infty$

$\Longrightarrow \lim_{x\to 3} \frac{1-x}{(x-3)^2} = -\infty$ because the one–sided limits are the same

$x = 3$ is a vertical asymptote to $y = \frac{1-x}{(x-3)^2}$.

Example $y = 2^{-1/x}$



From the graph:

$\lim_{x\to 0^-} 2^{-\frac{1}{x}} = +\infty$

$\lim_{x\to 0^+} 2^{-\frac{1}{x}} = 0$

$\Longrightarrow \lim_{x\to 0} 2^{-\frac{1}{x}}$ does not exist because the one–sided limits are not the same

$x = 0$ is a vertical asymptote to $y = 2^{-\frac{1}{x}}$.

Analytical Evaluation of Infinite Limits

The graphical method is often the best way to determine these limits. There is also an informal analytic method that seasoned math users often employ; they think

$$\frac{1}{\text{positive infinitesimal}} = \text{positive infinite number}, \quad \frac{1}{\text{negative infinitesimal}} = \text{negative infinite number}$$

or symbolically

$$\frac{1}{0^+} = +\infty \qquad \frac{1}{0^-} = -\infty$$

0^+ means a positive infinitesimal

$$a^+ = a + 0^+ \qquad a^- = a - 0^+$$

When we write $\frac{1}{0^+} = +\infty$, the left side is an infinite hyperreal number and the right side is the result of rounding it off. This is technically wrong, but everyone writes this and the meaning is clear; the reason for doing this is that $+\infty$ is an extended real number, but 0^+ is not accepted as one (why do you think it isn't?)

Example $\lim_{x\to 2^+} \frac{2^{-x}}{x-2} = \frac{2^{-2^+}}{0^+} = \frac{2^{-2}}{0^+} = +\infty$ (mixed real and hyperreal math, but rounding off result is correct)

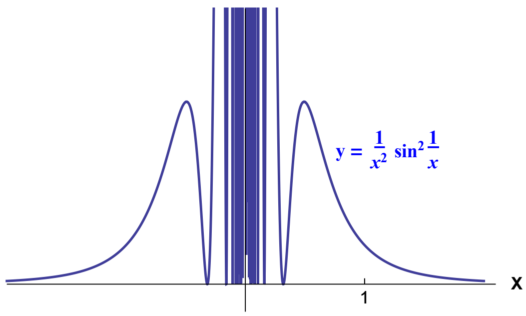
$\lim_{x\to 2^-} \frac{2^{-x}}{x-2} = \frac{2^{-2}}{0^-} = -\infty$

Therefore, because the two one-sided limits are not equal:

$\lim_{x\to 2} \frac{2^{-x}}{x-2}$ does not exist

Example $\lim_{x\to 0}\frac{1}{x^2}=\frac{1}{0^+}=+\infty$ because as $x\to 0^{\pm}$ (from either side), $x^2\to 0^+$.

Example $f(x)=\frac{1}{x^2}\sin^2\frac{1}{x}$



$\lim_{x\to 0}\frac{1}{x^2}\sin^2\frac{1}{x}$ does not exist because the function oscillates between 0 and $+\infty$ near $x=0$.

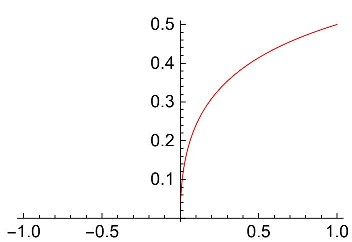
Exercises

In # 1 to 5, find the limits analytically using 0^+, etc. Check answers with graph provided.

#1. $\lim_{x\to 0^-}\frac{\sqrt{x}}{1+\sqrt{x}}$

$\lim_{x\to 0^+}\frac{\sqrt{x}}{1+\sqrt{x}}$

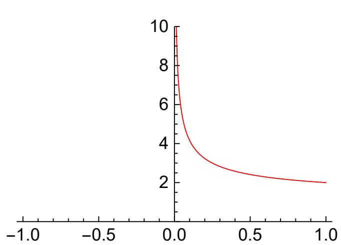
$\lim_{x\to 0}\frac{\sqrt{x}}{1+\sqrt{x}}$



#2. $\lim_{x\to 0^-}\frac{1+\sqrt{x}}{\sqrt{x}}$

$\lim_{x\to 0^+}\frac{1+\sqrt{x}}{\sqrt{x}}$

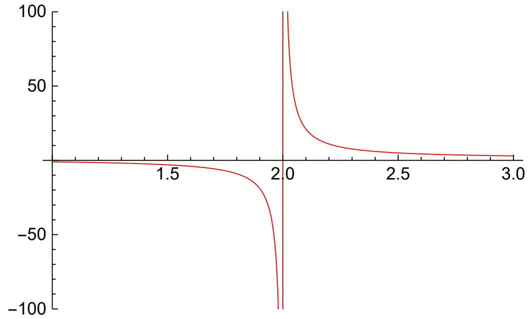
$\lim_{x\to 0}\frac{1+\sqrt{x}}{\sqrt{x}}$



#3. $\lim_{x \to 2^-} \frac{x}{x-2}$

$\lim_{x \to 2^+} \frac{x}{x-2}$

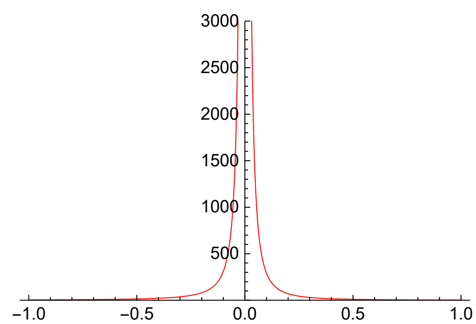
$\lim_{x \to 2} \frac{x}{x-2}$



#4. $\lim_{x \to 0^-} \frac{3-x}{x^2}$

$\lim_{x \to 0^+} \frac{3-x}{x^2}$

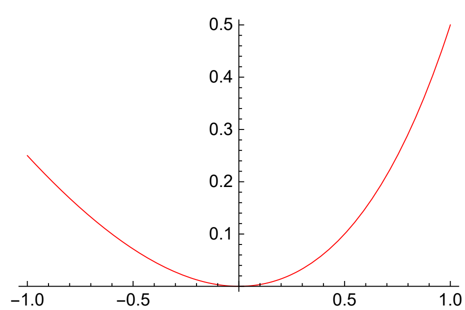
$\lim_{x \to 0} \frac{3-x}{x^2}$



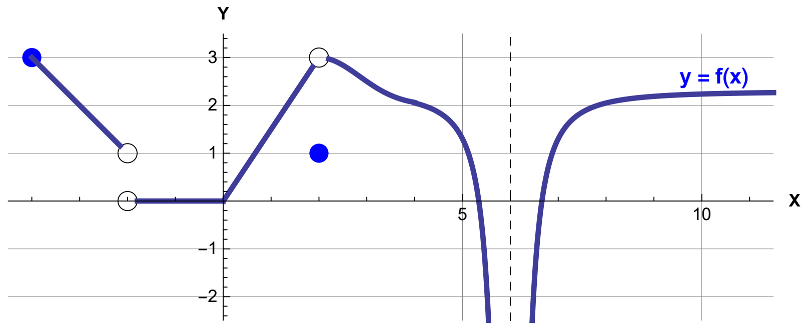
#5. $\lim_{x \to 0^-} \frac{x^2}{3-x}$

$\lim_{x \to 0^+} \frac{x^2}{3-x}$

$\lim_{x \to 0} \frac{x^2}{3-x}$



#6. y = f(x) is the function shown below.



a. $\lim_{x\to -2^-} f(x) =$ b. $\lim_{x\to 0^-} f(x) =$ c. $\lim_{x\to 2^-} f(x) =$ d. $\lim_{x\to 6^-} f(x) =$ e. $\lim_{x\to -4} f(x) =$

$\lim_{x\to -2^+} f(x) =$ $\lim_{x\to 0^+} f(x) =$ $\lim_{x\to 2^+} f(x) =$ $\lim_{x\to 6^+} f(x) =$ $\lim_{x\to 15} f(x) \doteq$

$\lim_{x\to -2} f(x) =$ $\lim_{x\to 0} f(x) =$ $\lim_{x\to 2} f(x) =$ $\lim_{x\to 6} f(x) =$

#7. $\lim_{x\to 0^+} \frac{1-x^x}{x}$

a. Work numerically.

b. Work graphically.

c. Plot the graph by computer.

#8. $\lim_{x\to 0^+} 3^{\frac{1}{x}}$

a. Work numerically.

b. Work graphically.

c. Plot the graph by computer.

Solutions

#1. DNE, 0, 0

#2. DNE, $+\infty$, $+\infty$

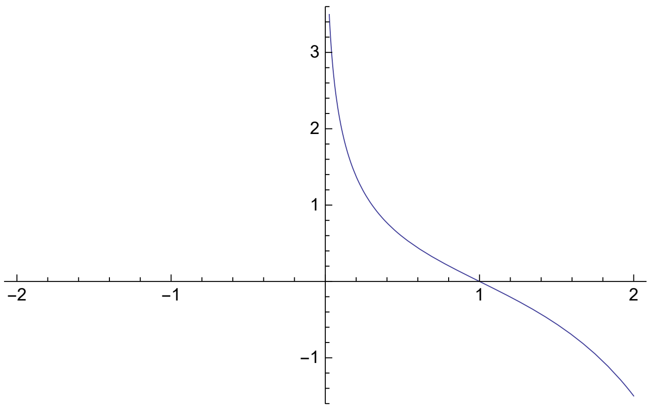
#3. $-\infty$, $+\infty$, DNE

#4. $+\infty$, $+\infty$, $+\infty$

#5. 0, 0, 0

#6.	a. 1	b. 0	c. 3	d. $-\infty$	e. 3 (endpoint of domain agreement)
	0	0	3	$-\infty$	2.4
	DNE	0	3	$-\infty$	

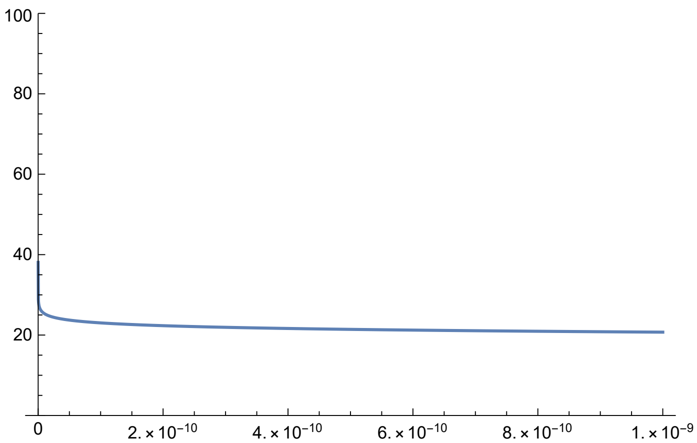
#7.




```
x := 1. × 10^-16
(1 - x^x) / x
```

```
36.6374
```

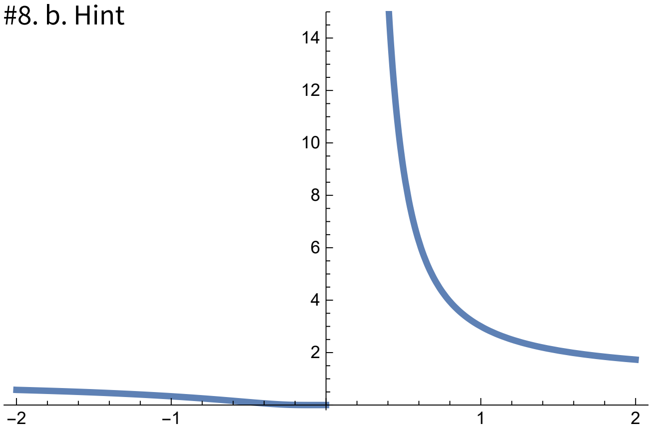
Explore this some more before you declare an answer.



? ? ?

My computer says $+\infty$. In the next calculus course you will learn how to do this analytically!

#8. b. Hint



1.6 Extensions of the Limit Idea. Limits at Infinity

It's time to review the new 'near equality' that will be be useful in doing theory, applications, and sustain a reasonably good mathematical work style later in your life. It applies to infinitesimal, finite hyperreal, and infinite number calculations. If $A \approx B$, but by mistake write $A = B$, you would be hyperreally wrong but still 'really' right! The main use of $\approx$ is to simplify expressions in order to extract the essence of a hyperreal expression. $\approx$ will be frequently used later in this course.

A Detailed Review of *Asymptotic Equality*

Definition A is ***asymptotically equal*** to B written $A \approx B$ means $\frac{A}{B} = 1 + \epsilon$ where ϵ is an infinitesimal.

Properties (proofs left as easy exercises)

1. $A \approx A$
2. $A \approx B \iff B \approx A$
3. $A \approx B, B \approx C \iff A \approx C$

Theorem $a \approx A, b \approx B \iff a \cdot A \approx b \cdot B$

Theorem $a \approx A, b \approx B \iff \frac{a}{A} \approx \frac{b}{B}$

Note: $A \approx 0$ is never true. Can you see why? This will never be a serious problem in calculus. The $\approx$ concept will be especially important when we do applications of integration

Examples Let dx be an infinitesimal, x+dx a finite hyperreal and X a positive infinite number.

Infinitesimal $3\,dx + dx^2 \approx 3\,dx$

because $\frac{3\,dx+dx^2}{3\,dx} = 1 + \frac{1}{3}\,dx = 1 + \epsilon.$

Finite Hyperreal $7 + dx^2 \approx 7$

because $\frac{7+dx^2}{7} = 1 + \frac{1}{7}\,dx^2 = 1 + \epsilon.$

Infinite Number $5X^3 - X^2 + 4 \approx 5X^3$

because $\frac{5X^3 - X^2 + 4}{5X^3} = 1 - \frac{5}{X} + \frac{4}{5X^3} = 1 + \epsilon.$

Note Eventually you get good at using $\approx$ to simplify calculations.

***With care*:**
Infinitesimals: keep dx, drop dx^2
Finite hyperreals: keep x, drop dx or dx^2
Infinite numbers: keep X^2, drop X or x or dx.

In applications, it is often obvious geometrically or physically or otherwise which terms can be ignored.

* * * * * * *

Limits at Infinity The limit idea can be extended to answer the question, "What is the behavior of the function when x is a large positive or negative number?" We answer this question by examining the function for x an infinite number and then rounding off.

Hyperreal Definition The limit of $f(x)$ as x approaches ***plus infinity*** is b, written

$$\lim_{x \to +\infty} f(x) = b$$

means $f(x) \approx> b$ for every positive infinite number x.

Hyperreal Definition The limit of $f(x)$ as x approaches ***minus infinity*** is b, written

$$\lim_{x \to -\infty} f(x) = b$$

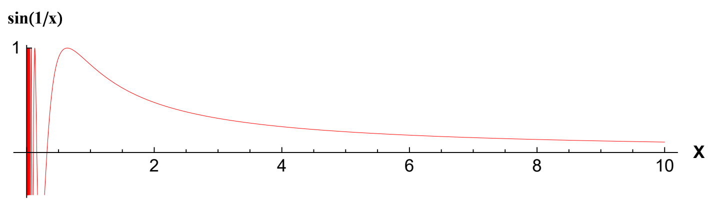
means $f(x) \approx> b$ for every negative infinite number x.

Example $\lim_{x \to +\infty} \sin \frac{1}{x} = 0$ because

$\sin \frac{1}{x}$

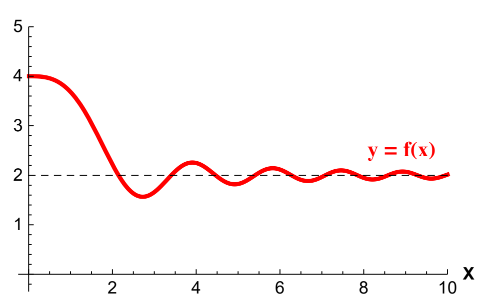
$\approx> \sin 0$ x is a positive infinite number $\Rightarrow \frac{1}{x}$ is a positive infinitesimal

$= 0.$



Definition If $\lim_{x \to +\infty} f(x) = b$ *or* $\lim_{x \to -\infty} f(x) = b$, we say that the line $y = b$ is a ***horizontal asymptote*** to the curve $y = f(x)$.

Example $f(x) = 2 + 2 \frac{\sin x^{3/2}}{x^{3/2}}$



From the graph, $\lim_{x \to +\infty} \left(2 + 2 \frac{\sin x^{3/2}}{x^{3/2}}\right) = 2$. So $y = 2$ is a horizontal asymptote. Note that a function ***may*** cross its horizontal asymptote (unlike a vertical asymptote) any number of times.

Limits of Rational Functions at Infinity by Analytic Methods

Examples A traditional method for a rational function is to divide the numerator and denominator by x raised to the degree of the denominator.

$$\lim_{x\to+\infty}\frac{x^2+2x+3}{x^3-5} = \lim_{x\to+\infty}\frac{\frac{x^2+2x+3}{x^3}}{\frac{x^3-5}{x^3}} = \lim_{x\to+\infty}\frac{\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}}{1-\frac{5}{x^3}} = \frac{0+0+0}{1-0} = \frac{0}{1} = 0$$

In the above calculation, we needed some more arithmetic for the symbols $+\infty$ and $-\infty$.

$$\frac{1}{+\infty} = 0^+ \qquad \frac{1}{-\infty} = 0^-$$

At the end of a calculation, the 'exponents' in 0^+ and 0^- are dropped, of course, because they are not extended reals.

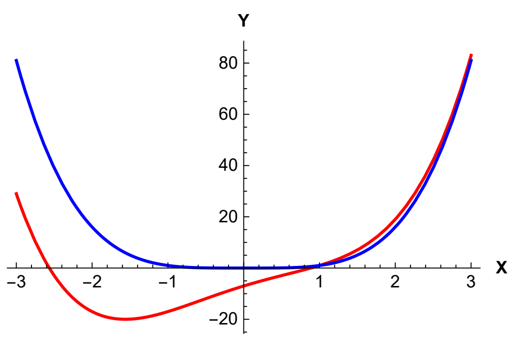
Quick Method The behavior of a polynomial function at infinity is determined by its leading term.

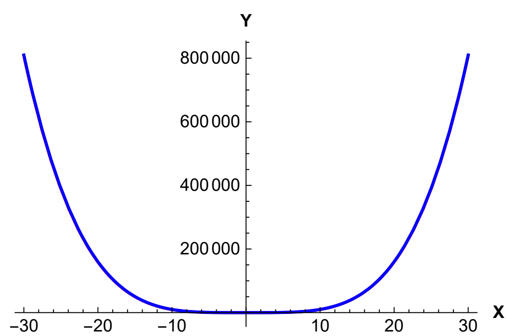
$$\lim_{x\to+\infty}(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_o) = \lim_{x\to+\infty} a_n x^n.$$

Proof

$$\lim_{x\to+\infty}(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_o) = \lim_{x\to+\infty} x^n\left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_o}{x^n}\right) = \lim_{x\to+\infty} a_n x^n \qquad (*)$$

Comment on the quick method For example, $x^4 - 2x^2 + 9x - 7$. When $x = 100$, the leading term x^4 is 100 million, but $2x^2$ is only 20,000, 9x a mere 900 and 7 barely counts. The graph of this function in red along with its leading term in blue are graphed below. In the slightly zoomed out picture on the right the graphs of the polynomial and its leading term are barely distinguishable. This means that you can ignore any terms other than the leading ones in rational function when taking the limit at infinity.





A polynomial and its leading term are nearly indistinguishable for x large.

Example

Quick Method Limit Method

$$\lim_{x\to+\infty}\frac{x^2+2x+3}{x^3-5} = \lim_{x\to+\infty}\frac{x^2}{x^3} = \lim_{x\to+\infty}\frac{1}{x} = 0$$

Quickest Method Asymptotically equality

$$\frac{x^2+2x+3}{x^3-5} \approx \frac{x^2}{x^3} = \frac{1}{x} \approx> 0 \text{ at infinity}$$

Examples $\{\frac{\infty}{\infty}\}$ indeterminate forms

$$\lim_{x\to+\infty}\frac{2x+1}{x^2+5x-4} = \lim_{x\to+\infty}\frac{2x}{x^2} = \lim_{x\to+\infty}\frac{2}{x} = 0$$

$$\lim_{x\to-\infty}\frac{2x^2+1}{x^2+5x-4} = \lim_{x\to-\infty}\frac{2x^2}{x^2} = \lim_{x\to-\infty} 2 = 2$$

$$\lim_{x\to+\infty}\frac{x^2+5x-4}{2x+1} = \lim_{x\to\infty}\frac{x^2}{2x} = \lim_{x\to\infty}\frac{x}{2} = +\infty$$

Examples The quick methods often, with care, work well with other limits involving fractions. (Of course, you must be aware of any indeterminate forms that may occur in the process and treat them correctly.)

$$\lim_{x\to+\infty}\frac{2\sqrt{x}+1}{x^2+5x-4} = \lim_{x\to+\infty}\frac{2\sqrt{x}}{x^2} = \lim_{x\to+\infty}\frac{2}{x^{3/2}} = 0.$$

$$\lim_{x\to+\infty}\frac{2x^2+\sin x}{x^2+5x-4} = \lim_{x\to+\infty}\frac{2x^2}{x^2} = 2$$

because x^2 grows much more rapidly than $\sin x$ as $x \to +\infty$.

Using asymptotic thinking the previous example can be written

$\frac{2x^2+\sin x}{x^2+5x-4} \approx \frac{2x^2}{x^2} = 2$ at infinity.

$$\lim_{x\to+\infty}\sqrt{x^2-6x+3} - x \quad \{\infty - \infty\}$$

$$= \lim_{x\to+\infty}\left(\sqrt{x^2-6x+3} - x\right)\cdot\frac{\sqrt{x^2-6x+3} + x}{\sqrt{x^2-6x+3} + x} \qquad \text{rationalizing numerator}$$

$$= \lim_{x\to+\infty}\frac{x^2-6x+3 - x^2}{\sqrt{x^2-6x+3} + x}$$

$$\approx \lim_{x\to+\infty}\frac{-6x+3}{\sqrt{x^2}+x} \qquad \text{quick method}$$

$$= \lim_{x\to+\infty}\frac{-6x}{x+x} = \lim_{x\to+\infty}\frac{-6x}{2x} = -3 \qquad \text{quick method, } x > 0$$

Caution Never use the methods for limits at infinity for other limits.

$\lim_{x\to0}\frac{2x^2+1}{x^2+5x-4} = \lim_{x\to0}\frac{2x^2}{x^2} = \lim_{x\to0} 2 = 2$ is wrong. The correct answer is $-\frac{1}{4}$, of course.

Final comments Limit notation is preferable because most math users are familiar with it. Hyperreal thinking is preferable because it tends to be better focused, namely the value of the limit as $x\to a$ is found by examining the values of f infinitesimally close to $x=a$; it's better than messing around with not very close real numbers. Of course, if you are doing approximate limits you will use real numbers, but hyperreal thinking gives you perspective about the process.

Important Note about Limits

The result of any limit calculation can only be one of the following:

1. An extended real number:
   - a real number
   - the symbol $+\infty$ or $-\infty$
2. Does not exist.

Calculus includes any topic that involves a limit calculation. So a result in calculus can **only** be one of the above two outcomes. When we do calculus, we will always follow this rule because it is generally meaningful to do so. Some calculus textbooks do not allow $+\infty$ or $-\infty$ for some calculus computations. Also, we will adopt the convention that a limit **exists** at a domain endpoint if the appropriate one-sided limit exists; again, not all mathematicians agree with this; however, most engineers and scientists do because the results **always** have a reasonable interpretation.

Exercises

#1. $\lim_{x\to+\infty} \frac{1+\sqrt{x}}{\sqrt{x}}$

$\lim_{x\to-\infty} \frac{1+\sqrt{x}}{\sqrt{x}}$

#2. $\lim_{x\to+\infty} \frac{x}{x-2}$

$\lim_{x\to-\infty} \frac{x}{x-2}$

#3. $\lim_{x\to+\infty} \frac{x^2}{3-x}$

$\lim_{x\to-\infty} \frac{x^2}{3-x}$

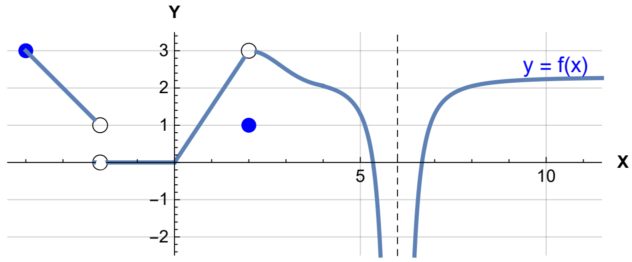
$\lim_{x\to+\infty} \frac{3-x}{x^2}$

$\lim_{x\to-\infty} \frac{3-x}{x^2}$

#4. $\lim_{x\to+\infty} \left(x - \sqrt{x^2 - 6x}\right)$

$\lim_{x\to-\infty} \left(x - \sqrt{x^2 - 6x}\right)$

#5. $y = f(x)$ is the function shown below.



a. $\lim_{x\to -2^-} f(x) =$ $\lim_{x\to -2^+} f(x) =$ $\lim_{x\to -2} f(x) =$

b. $\lim_{x\to 0^-} f(x) =$ $\lim_{x\to 0^+} f(x) =$ $\lim_{x\to 0} f(x) =$

c. $\lim_{x\to 2^-} f(x) =$ $\lim_{x\to 2^+} f(x) =$ $\lim_{x\to 2} f(x) =$

d. $\lim_{x\to 6^-} f(x) =$ $\lim_{x\to 6^+} f(x) =$ $\lim_{x\to 6} f(x) =$

e. $\lim_{x\to -\infty} f(x) =$ $\lim_{x\to +\infty} f(x) =$ $\lim_{x\to -4} f(x) =$

#6. Verify the proof of Equation (*).

#7. Work each carefully. Check each using the quick method (preferably doing mentally).

$\lim_{x\to -\infty} \dfrac{2x+3}{x^3-3x+5}$ $\lim_{x\to -\infty} \dfrac{2x^3+3x^2}{x^3-3x+7}$ $\lim_{x\to -\infty} \dfrac{2x^4+3}{x^3-3x+5}$

$\lim_{x\to +\infty} \dfrac{2x^5+3}{x^3-3x+5}$ $\lim_{x\to +\infty} \dfrac{2x^2+3x^3}{x^3-3x+7}$ $\lim_{x\to +\infty} \dfrac{3-2x+3^x}{x^3-3x+5}$

$\lim_{x\to -\infty} \dfrac{2x^5+3}{x^3-3x+5}$ $\lim_{x\to -\infty} \dfrac{2x^2-3x^3}{x^3-3x+7}$ $\lim_{x\to -\infty} \dfrac{3-2x^4}{x^3-3x+5}$

#8. $\lim_{x\to +\infty} 2^{\frac{1}{x}}$

$\lim_{x\to -\infty} 2^{\frac{1}{x}}$

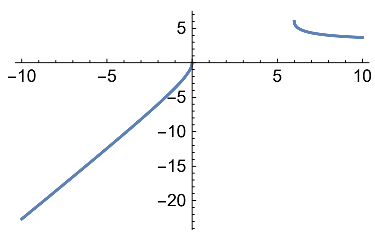
Solutions

#1. 1, DNE

#2. 1, 1

#3. - ∞, +∞, 0, 0

#4. 3, - ∞. Hints: rationalize numerator and use $\sqrt{x^2} = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$



#5. a. 1   b. 0   c. 3   d. −∞   e. DNE

0   0   3   −∞   ≐ 2.5

DNE   0   3   −∞   3

#7. 0   2   +∞

0   2   −∞

+∞   3   +∞

+∞   −3   +∞

1.7 Application: Graphing Rational Functions

Many students encounter rational functions early in university applied courses. Asymptotes plus a few data points give a quick way of graphing them fairly accurately. Of course, if you need a completely accurate graph, you would use a computer graphing utility and in real life coefficients are usually not integers and numerical factoring is required. But graphing by hand gives you important insights into the behavior of these functions.

Rational Functions Review

$$y = \frac{P(x)}{Q(x)} = \frac{a_m x^m + \cdots}{a_n x^n + \cdots},\quad m \text{ and } n \text{ non-negative integers}$$

Vertical Asymptotes

Where $Q(x) = 0$

Horizontal Asymptotes

$\deg P < \deg Q \implies y = 0$ is a horizontal asymptote

$\deg P = \deg Q \implies y = \frac{a_m}{a_n}$ is a horizontal asymptote

$\deg P > \deg Q \implies$ a slant or curved asymptote of degree x^{m-n}

In preparation for graphing, it is often helpful to factor the denominator in order to determine the vertical asymptotes. Factoring the numerator is useful if you wish to know the zeros of the rational function.

If $\deg P > \deg Q$, you will want to divide $P(x)$ by $Q(x)$ to determine the slant or curved asymptote.

The Method

1. **Find all asymptotes and draw them as a dashed curves on a graph.**
2. **Then find a few well chosen data points to 'nail down' the curve. Place them on the graph.**
3. **Sketch the curve taking into account the above information.**

Normally the result is quite good considering the small amount work required. Occasionally there is a surprise 'wiggle' for which you need some calculus information that you will learn in Chapters 2 and 3.

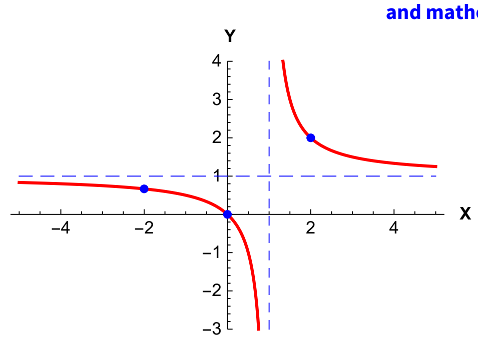
Example $y = \frac{x}{x-1}$

Vertical asymptote: $x = 1$

Horizontal asymptote: $y = \frac{x}{x-1} \approx 1$ at infinity

x	$y = \frac{x}{x-1}$
-2	$\frac{2}{3}$
0	0
2	2

Choice of points requires a combination of artistic and mathematics skills.

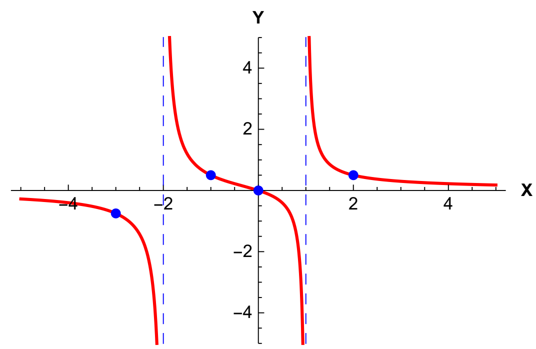


Example $y = \frac{x}{x^2 + x - 2} = \frac{x}{(x-1)(x+2)}$

Vertical asymptote: x = -2, 1

Horizontal asymptote: $\mathbf{y} = \frac{x}{x^2 + x - 2} \approx \frac{x}{x^2} = \frac{1}{x} \approx> 0$

x	$y = \frac{x}{x-1}$
-3	$-\frac{3}{5}$
-1	$\frac{1}{2}$
0	0
2	$\frac{1}{2}$



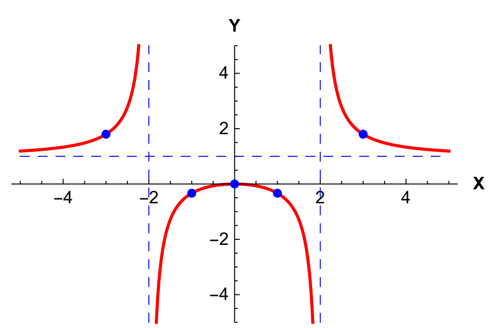
Example $y = \frac{x^2}{x^2 - 4} = \frac{x^2}{(x-2)(x+2)}$

Vertical asymptote: x = -2, 2

Horizontal asymptote: $\mathbf{y} = \frac{x^2}{x^2 - 4} \approx \frac{x^2}{x^2} \approx> 1$

x	$y = \frac{x}{x-1}$
-3	$\frac{9}{5}$
-1	$-\frac{1}{3}$
0	0
1	$-\frac{1}{3}$
3	$\frac{9}{5}$

See if you can discover the rules for when a rational function at a vertical asymptote is on the same side of the x-axis or not.



Example **Slant Asymptote**

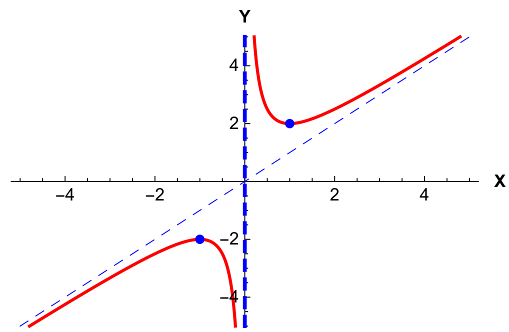
$$y = \frac{x^2+1}{x} = x + \frac{1}{x}$$

Vertical asymptote: $x = 0$

By monomial division

$y = x + \frac{1}{x}$. The slant asymptote is $y = x$ since $\frac{1}{x} \approx> 0$ for x infinite.

x	$y = \frac{x^2+1}{x}$
−1	−1
1	1



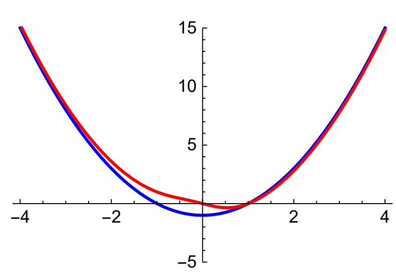
Example **Curved Asymptote** You verify the details.

$$y = \frac{x^4 - x}{x^2+1} = -1 + x^2 + \frac{1-x}{1+x^2}$$

By computer or long division

$y = -1 + x^2 + \frac{1-x}{1+x^2}$. The curved asymptote is the parabola

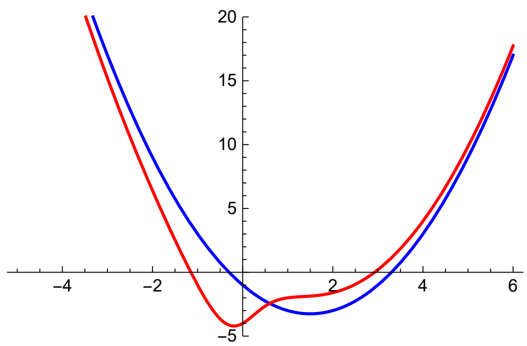
$y = -1 + x^2$ since $\frac{1-x}{1+x^2} \approx> 0$ for $x = \pm\infty$.



Example **Curved Asymptote** You verify the details.

$$y = \frac{x^4 - 3x^3 + 2x - 4}{x^2 + 1}$$

$y = -1 - 3x + x^2 + \dfrac{-3 + 5x}{1 + x^2}$. The curved asymptote is $y = -1 - 3x + x^2$



Exercises

For each question

a. Find all vertical, horizontal, slant and curved asymptotes. Graph.

b. Make a short table of test points.

c. Use the above to sketch a good graph.

1. $y = \frac{1}{x-2}$
2. $y = \frac{x}{2x-1}$
3. $y = \frac{x}{x-2}$
4. $y = \frac{2x}{(x+1)(x-2)}$
5. $y = \frac{x^2}{(x-2)^2}$
6. $y = \frac{x^3}{x-2}$
7. $y = \frac{x^3+1}{x-2}$
8. $y = \frac{x^4}{(x+2)(x-2)}$

Note again On occasion this method can miss an important detail such as a low point. Calculus will help you with such problems later.

Solutions are not provided as they would make the exercises trivial.
If you wish, graph with ***Wolfram Alpha*** for a check on your solutions.

Chapter 2 The Derivative

2.0 We Need (something called) the Derivative

In this section we look at some applications that require a calculation involving limits. It gives the growth rate of the function. This calculation will be called ***the derivative*** because it is *derived* from the function under investigation.

We finally made it to the calculus!

Real Numbers

⟶ Algebra

⟶ Functions

⟶ Continuity & Limits

⟶ Calculus!

The Derivative

The Definite Integral

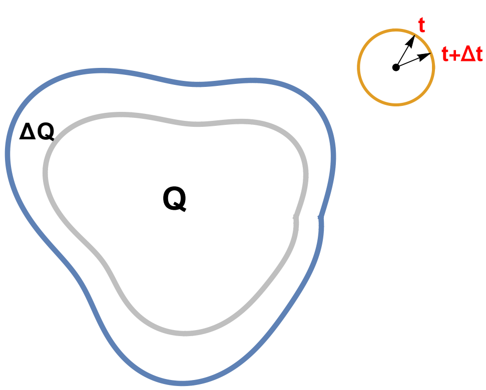
The Instantaneous Growth Rate In high school you used functions to describe the size of a quantity Q at a time t. Just as important may be finding its growth rate at time t.

The average growth rate of Q on the interval from time t to time t+Δt is

$$r_{av} = \frac{\Delta Q}{\Delta t}.$$

(Δ is the upper case Greek letter delta. Δt means the change in t and ΔQ means the corresponding change in Q. This use of Δ is common in mathematics.)

This formula says that the average growth rate is proportional to the change in the quantity and inversely proportional to the change in time; a smaller Δt for the same ΔQ produces a larger growth rate, which is reasonable.



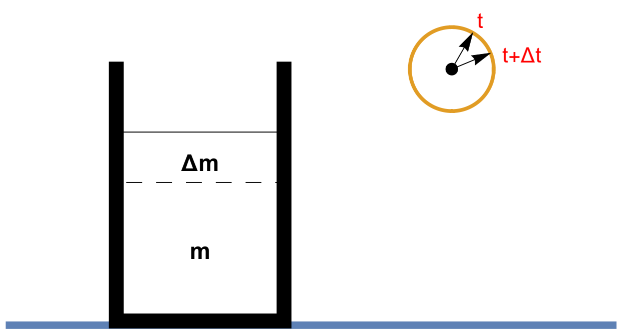
In many applications we are not really interested in this average rate of change over the interval from t to t+Δt, but rather the (instantaneous) rate of change r ***at*** time t. Unfortunately, we cannot just set $\Delta t = 0$ which implies that $\Delta Q = 0$ because we would get the growth rate at time t

$$r(t) = \frac{0}{0}$$

which is not a defined number (it always requires two distinct time measurements for a rate calculation). What we must do is start with a non-zero Δt and let Δt shrink to 0 without ever letting Δt equal to 0; that is, we find the *limit* as Δt approaches 0 of $\frac{\Delta Q}{\Delta t}$ and write for the instantaneous growth rate

$$\boxed{r(t) = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t}}.$$

Example 1 The amount of bacteria in a culture is $m = f(t) = 1.3^t$ mg, t in hours. Find the growth rate when $t = 10$ hours.



$$r(10) = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(10+\Delta t) - f(10)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1.3^{10+\Delta t} - 1.3^{10}}{\Delta t}$$

If we let $\Delta t = 0$, we would get $\frac{f(10+0) - f(10)}{0} = \left\{\frac{0}{0}\right\}$, which is not a number.
Let us evaluate this limit *numerically* (i.e., approximately) by examining the quotient for a sequence of Δt's approaching 0.

Δt	$\frac{1.3^{10+\Delta t} - 1.3^{10}}{\Delta t}$
1	4.13575
.1	3.66478
.01	3.62166
.001	3.61739
.0001	3.61696
↓	↓
0	**?**

It looks like the growth rate after 10 hours is

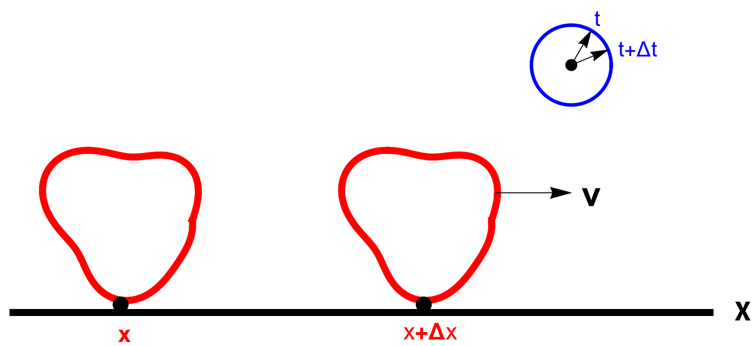
$$r(10) = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} \doteq 3.617 \frac{\text{mg}}{\text{hour}}.$$

This means that at time 10 hours you expect the mass to increase by about 3.617 mg during the next hour.

Velocity from Distance Find the velocity of a particle whose position is given by $x = f(t)$. (It is the rate at which the distance is changing at time t.)

The average velocity of the particle on the interval from time t to time $t+\Delta t$ is

$$\mathbf{v_{av}} = \frac{\Delta \mathbf{x}}{\Delta \mathbf{t}}.$$



In many applications we are not really interested in this average velocity over the interval from t to $t+\Delta t$, but rather the (instantaneous) velocity v ***at*** time t. For example, a policeman would never say, "I clocked your average speed on the interval from 9:45 to 9.46 AM to be $125 \frac{\text{Km}}{\text{hour}}$," but rather, "Your speed ***at*** 9:45 was $130 \frac{\text{Km}}{\text{hour}}$." Unfortunately, we cannot just set $\Delta t = 0$ (which implies $\Delta x = 0$) because we would get the velocity at time t to be $v(t) = \frac{0}{0}$ which is not a number (it always requires two distinct time measurements for a velocity calculation.)

What we must do is start with a non-zero Δt and let Δt shrink to 0 without ever letting Δt equal to zero. Again we write

$$\boxed{v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}}.$$

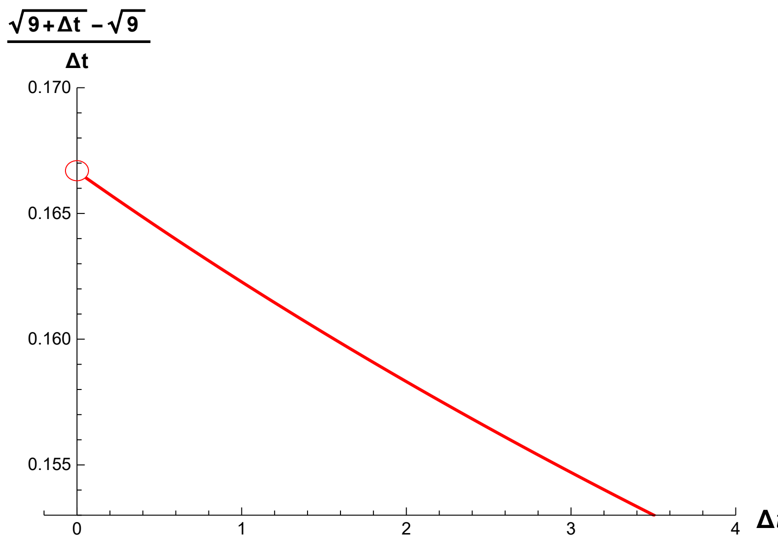
Example 2 The distance of a cart moving along the X-axis is given by $x = \sqrt{t}$ cm, t in seconds. Find the velocity when $t = 9$ seconds.

$$v(9) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(9+\Delta t) - f(9)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\sqrt{9+\Delta t} - \sqrt{9}}{\Delta t}$$

Let us evaluate this is *graphically* by graphing this quotient against Δt and see what its value is close to $\Delta t = 0$ on the right.



It looks like the velocity at 9 seconds is

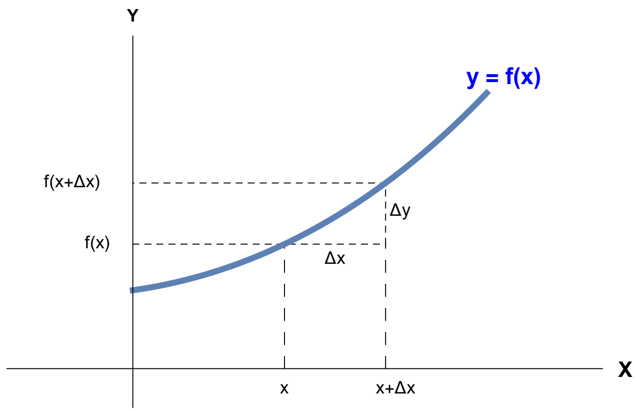
$$v(9) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \doteq 0.167 \frac{\text{cm}}{\text{second.}}.$$

This means that at time $t = 9$ seconds it looks like the cart will travel about 0.167 cm during the next second.

Slope from Height Find the slope m of the curve y = f(x) at the point x.

The average slope of the curve on the interval from x to x+Δx is

$$m_{av} = \frac{\Delta y}{\Delta x}.$$



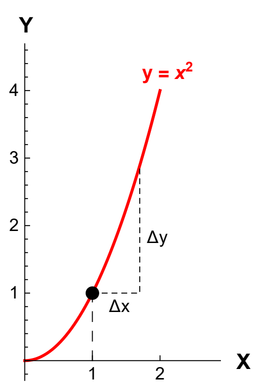
Again we want the slope at x, not the average slope on the interval from x to x+Δx. Again we cannot just let Δx = 0 because we would get

$$m(x) = \frac{0}{0}$$

which is not a number (it always requires two distinct points for a slope calculation.) What we must do is start with a non-zero Δx and let Δx shrink to 0 without ever letting Δx equal to zero. We write

$$\boxed{m(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}.$$

Example 3 Find the slope of the curve y = f(x) = x^2 at x = 1.



$$m(1) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta t \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x}$$

$$= \lim_{\Delta t \to 0} \frac{(1+\Delta x)^2 - 1^2}{\Delta x}$$

Let us evaluate this limit *analytically*.

$= \lim_{\Delta x \to 0} \frac{(1+2\,\Delta x+\Delta x^2)-1}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\Delta x(2+\Delta x)}{\Delta x}$

$= \lim_{\Delta x \to 0} (2+\Delta x)$ — We can cancel because we do not allow $\Delta x = 0$.

$= 2$ — Because when Δx is close to 0, $2+\Delta x$ is close to 2.

So the slope at $x = 1$ is

$m(1) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ — Assuming distance is measured in meters.

$= 2\,\frac{m}{m}$

$= 2$

This means that at the point (1,1) it looks like if you go one unit to the right, y increases by about two units.

You are now using limits. So you are doing calculus, the part called ***derivative*** calculus. In the rest of this chapter you will get proficient at derivative calculus. Three equivalent styles according to the user are given for the definition of derivative.

Definition of Derivative The *derivative* of the function f at x = a

Hyperreal Version $\frac{dy}{dx} = \frac{f(a+dx)-f(a)}{dx} \approx> f'(a)$

Pure Math Version $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Rough Applied Version $\frac{dy}{dx} = \frac{f(a+dx)-f(a)}{dx} \approx f'(a)$ or $= f'(a)$

Memorize these ASAP!

Exercises

Reread this section thoughtfully. Work each of the following using the styles of the examples. State the units in your answer. Note when a certain style does not work.

1. The mass of a melon is given by $m = f(t) = 3t^2$ gm, t in weeks. Find its growth rate when $t = 10$ weeks. Evaluate the limit analytically.

2. The position of a particle moving along the X-axis is given by $x = g(t) = 2t + 1$ cm, t in seconds. Find its (instantaneous) velocity when $t = 3$ seconds. Evaluate the limit graphically.

3. A curve is given by $y = k(x) = \frac{4}{x}$. Find its slope when $x = 2$. Evaluate the limit numerically.

Solutions

1. $m = f(t) = t^2$ gm, t in weeks at 10 weeks. Do analytically.

$$\frac{dm}{dt} = \frac{f(10+dt) - f(10)}{dt}$$ hyperreal style

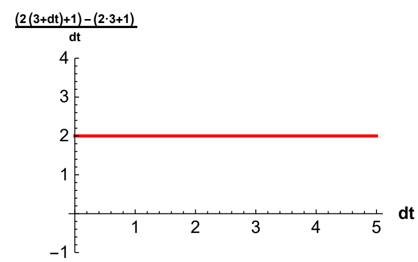
$$= \frac{(10+dt)^2 - 10^2}{dt}$$
$$= \frac{(100 + 20\,dt + dt^2) - 100}{dt}$$
$$= \frac{20\,dt + dt^2}{dt}$$
$$= \frac{t(20 + dt)}{dt}$$
$$= 20 + dt$$
$$\approx> 20\ \frac{\text{gram}}{\text{week}}$$

2. $x = g(t) = 2t + 1$ cm, t in seconds at 3 seconds. Do graphically.

$$\frac{dx}{dt} = \frac{g(3+dt) - g(3)}{dt}$$ hyperreal style

$$= \frac{(2\,(3+dt)+1) - (2\cdot 3+1)}{dt}$$

Note Using the hyperreal variable dt as a real variable is poor ascethetics. ⟶



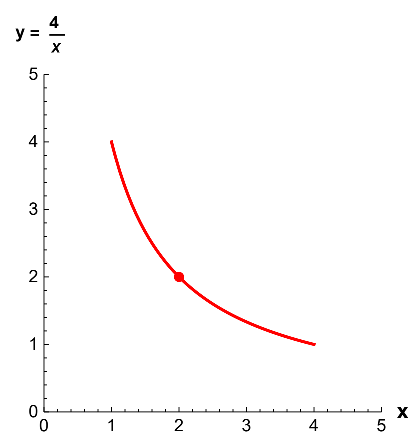
Looks like the velocity at 3 seconds is $2\ \frac{\text{cm}}{\text{second}}$.

Note A stupid, but correct way to work this easy problem.

3. $y = k(x) = \frac{4}{x}$. Find the slope at $x = 2$. Do numerically.

$$\frac{dy}{dx} = \frac{\frac{4}{2+dx} - \frac{4}{2}}{dx}$$

dx	$\frac{\frac{4}{2+dx} - \frac{4}{2}}{dx}$
1	-0.66666
0.1	-0.95238
0.01	-0.99550
0.001	-0.99950
↓	↓
0	**-1**



Answer: the slope at $x = 2$ is $m = -3$.

2.1 The Derivative. Starting out

Let us look at the equivalent definitions of derivative again.

Definition of Derivative The *derivative* of the function f at x = a:

Hyperreal Version $\frac{dy}{dx} = \frac{f(a+dx) - f(a)}{dx} \approx> f'(a),\ dx \neq 0$

Pure Math Version $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Rough Applied Version $\frac{dy}{dx} = \frac{f(a+dx) - f(a)}{dx} \approx f'(a)$

Notes:

The **hyperreal definition** is preferred for doing proofs. It's symbol dy/dx says the (pre)derivative at the p oint x = a is the ratio of the change in y to the change in x there. The closest real number to dy/dx gives the derivative f '(a). **≈>** is a clear, often easy operation which associates a hyperreal number to the closest real number (more about this later).

In the **pure math definition**, f '(a) is a convenient way of indicating the derivative at a point x = a and is a more concise notation than $\left.\frac{dy}{dx}\right|_{x=a}$ but fails in indicating the meaning of the derivative.

The **applied version** is convenient, user friendly and naturally used by many applied calculus users. It is not wrong, but **≈>** is more descriptive.

Yes, you may use whichever style of derivative definition you prefer. In mathematics, when doing theory) we will usually use the hyperreal form.

Finding the derivative by the definition It is a tradition to require beginning calculus students to do a few of these. It is just as easy to find the derivative at any point x in the domain as at a particular point x = a: so we will usually do that. All these are indeterminate forms of the type $\left\{\frac{0}{0}\right\}$ if we illegally allowed dx or Δx to be 0.

Example A polynomial function $y = x^2 - 3x + 2$.

$\frac{dy}{dx} = \frac{[(x+dx)^2 - 3(x+dx) + 2] - (x^2 - 3x + 2)]}{dx}\ \left\{\frac{0}{0}\right\}$ definition of derivative

$= \frac{[x^2 + 2x\,dx + dx^2 - 3x - 3\,dx + 2] - [x^2 - 3x + 2]}{dx}$ expanding

$= \frac{2x\,dx - 3\,dx + dx^2}{dx}$ or $\approx \frac{2x\,dx - 3\,dx}{dx}$ simplifying

$= \frac{dx(2x - 3 + dx)}{dx}$ factoring

= 2x - 3 + dx can cancel: dx ≠ 0

≈> 2x - 3 (There is no f(x), so don't use the f '(x) notation.)

Example A rational function $y = \frac{1}{x}$.

$$\frac{dy}{dx} = \frac{1}{dx}\left(\frac{1}{x+dx} - \frac{1}{x}\right) \quad \left\{\frac{0}{0}\right\}$$
$$= \frac{1}{dx}\left(\frac{x-(x+dx)}{x(x+dx)}\right)$$
$$= \frac{-dx}{dx\,x(x+dx)}$$
$$= \frac{-1}{x(x+dx)}$$
$$\approx> -\frac{1}{x^2}$$

Example An algebra function $y = \sqrt{x}$.

$$\frac{dy}{dx} = \frac{\sqrt{x+dx} - \sqrt{x}}{dx} \quad \left\{\frac{0}{0}\right\} \qquad \text{definition of derivative}$$
$$= \frac{\sqrt{x+dx} - \sqrt{x}}{dx}\,\frac{\sqrt{x+dx} + \sqrt{x}}{\sqrt{x+dx} + \sqrt{x}} \qquad \text{multiply by } 1$$
$$= \frac{(x+dx) - x}{dx\left(\sqrt{x+dx} + \sqrt{x}\right)} \qquad (a - b)(a + b)$$
$$= \frac{1}{\sqrt{x+dx} + \sqrt{x}}$$
$$\approx> \frac{1}{2\sqrt{x}}$$

The Power Rule *An important derivative formula*. Formulas make calculus productive.

The '***operator notation***' $\frac{d}{dx}$ means 'take the derivative of what follows.'

$$\mathbf{\frac{d}{dx}(x^n) = n\,x^{n-1}}$$

n a positive integer.

To do the proof, we need some algebra formulas.

Difference of Powers formula $a^n - b^n$

$a^2 - b^2 = (a - b)\,(a + b)$
$a^3 - b^3 = (a - b)\,(a^2 + a\,b + b^2)$
$a^4 - b^4 = (a - b)\,(a^3 + a^2\,b + a\,b^2 + b^3)$
$\vdots$
$a^n - b^n = (a - b)\,(a^{n-1} + a^{n-2}\,b + \cdots + a\,b^{n-2} + b^n)$ there are n+1 terms inside the last parentheses

Let us try one for practice with n = 5.

$a^5 - b^5 = (a - b)\,(a^4 + a^3\,b + a^2\,b^2 + a\,b^3 + b^4)$

Derivation of the Power Rule

$\frac{d}{dx}(x^n)$

$= \frac{(x+dx)^n - x^n}{dx}$ definition of derivative

$= \frac{1}{dx}[(x+dx) - x][(x+dx)^{n-1} + (x+dx)^{n-2}x + (x+dx)^{n-3}x^2 + \cdots + (x+dx)x^{n-2} + x^{n-1}]$ Let a = x+dx, b = x

$= \frac{1}{dx}dx[(x+dx)^{n-1} + (x+dx)^{n-2}x + (x+dx)^{n-3}x^2 + \cdots + (x+dx)x^{n-2} + x^{n-1}]$

$= (x+dx)^{n-1} + (x+dx)^{n-2}x + (x+dx)^{n-3}x^2 + \cdots + (x+dx)x^{n-2} + x^{n-1}$

$\approx> x^{n-1} + x^{n-2}x + x^{n-3}x^2 + \cdots + x\,x^{n-2} + x^{n-1}$ n equal terms

$= nx^{n-1}$

We only proved the power formula for n a positive integer. Will show later it is true for n any real number. We will allow ourselves to use the power rule for any n now.

Examples

$\frac{d}{dx}(x^3) = 3x^2$

$\frac{d}{dx}(x^{100}) = 100x^{99}$ don't try this by the definition at home!

$\frac{d}{dx}(\frac{1}{x^2}) = \frac{d}{dx}(x^{-2}) = -2x^{-3}$

$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1//2}) = \frac{1}{2}x^{-1/2}$

$\frac{d}{dx}(x^\pi) = \pi x^{\pi-1}$

$\frac{d}{dt}(t^5) = 5t^4$

$\frac{d}{dx}(\pi^2) = 0$

Tangent and perpendicular lines to a curve at a point

Tangent lines are an elementary application of derivatives important for understanding the derivative and for applications the derivative.

The tangent line to the curve y = f(x) at the point (x_1, y_1) with slope m is

$y = y_1 + m(x - x_0)$,

where $m = f'(x_1)$. Then the tangent line formula is

$$\boxed{y = f(a) + f'(a)(x - a)}$$

The perpendicular line to the curve y = f(x) at the point (x_1, y_1) with slope m is

$$\boxed{y = f(a) - \frac{1}{f'(a)}(x - \mathrm{a})}$$

Higher Order Derivatives

In this context, if $y = f(x)$, $\frac{dy}{dx} = f'(x)$ is called the ***first derivative***. Since it is also a function, you can take its derivative:

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$, the ***second derivative*** (Why did we write d^2 but dx^2?)

and so on. Common alternative notions:

$y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}, \ldots$

$f(x), f'(x), f''(x), f'''(x), f^{(4)}(x), \ldots, f^{(n)}(x), \ldots$

Others are

Dy, $D_x y$, y' and $\dot{y}$ used in some applications.

Exercises

1. Use the definition of derivative to find
   a. $\frac{d}{dx}\left(x^2 + 5x + 3\right) =$
   b. $\frac{d}{dx}\left(\frac{2}{x-3}\right) =$
   c. $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) =$
   d. $\frac{d}{dx}(c) =$
   e. $\frac{d}{dx}(mx + b) =$

2. Harder ones, by the definition of derivative.
   a. $\frac{d}{dx}\left(x^4\right) =$
   b. $\frac{d}{dx}\left(\frac{x}{x^2+1}\right) =$
   c. $\frac{d}{dx}\left(\sqrt[3]{x}\right) =$

3. What does the word mnemonic mean? How do you pronounce it.

4. Work using the Power Rule.
   a. $\frac{d}{dx}\left(x^7\right) =$
   b. $\frac{d}{dx}\left(x^{\sqrt{3}}\right) =$
   c. $\frac{d}{dx}\left(\sqrt[3]{x}\right) =$

5. a. Find the tangent line and perpendicular line to $y = f(x) = x^2$ at $x = 2$.
   b. Graph and discuss the result.

6. Find the second derivative of each function in #1.

Solutions

1. a. $\frac{d}{dx}(x^2 + 5x + 3)$

$= \frac{[(x+dx)^2+5(x+dx)+3]-[x^2+5x+3]}{dx}$

$= \frac{x^2+2x\,dx+dx^2+5x+5\,dx+3-x^2-5x-3}{dx}$

$= \frac{dx(2x+dx+5)}{dx}$

$\approx> 2x + dx + 5$

b. $\frac{d}{dx}\left(\frac{2}{2x+3)}\right)$

$= \frac{1}{dx}\left[\frac{2}{2(x+dx)+3)} - \frac{2}{2x+3)}\right]$

$= \cdots$

$= -\frac{4}{(2x+3)^2}$

c. $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)$

$= \frac{1}{dx}\left[\frac{1}{\sqrt{x+dx}} - \frac{1}{\sqrt{x}}\right]$

$= \frac{1}{dx}\left[\frac{\sqrt{x} - \sqrt{x+dx}}{\sqrt{x+dx}\,\sqrt{x}}\right]$

$= \frac{1}{dx}\left[\frac{\sqrt{x} - \sqrt{x+dx}}{\sqrt{x+dx}\,\sqrt{x}} \cdot \frac{\sqrt{x} + \sqrt{x+dx}}{\sqrt{x} + \sqrt{x+dx}}\right]$ $(a-b)(a+b) = a^2 - b^2$

$= \frac{1}{dx}\left[\frac{x - (x+dx)}{\sqrt{x+dx}\,\sqrt{x}\,\left(\sqrt{x} + \sqrt{x+dx}\right)}\right]$

$= \frac{1}{dx}\left[\frac{-dx}{\sqrt{x+dx}\,\sqrt{x}\,\left(\sqrt{x} + \sqrt{x+dx}\right)}\right]$

$= \frac{-1}{\sqrt{x}\,\sqrt{x}\,\left(\sqrt{x} + \sqrt{x+dx}\right)}$

$\approx> \frac{-1}{\sqrt{x}\,\sqrt{x}\,\left(\sqrt{x} + \sqrt{x}\right)}$

$= \frac{-1}{2x^{3/2}}$

d. 0

e. m

2 a. $4x^3$

b. $\frac{1-x^2}{(x^2+1)^2}$

c. Hint: $(a-b)(a^2+ab+b^2) = a^3 - b^3$

3. See dictionary.

4. a. $7x^6$

b. $\sqrt{3}\,x^{\sqrt{3}-1}$

c. $\frac{1}{3x^{2/3}}$

#5. a. Tangent line:

$f(2) = 2^2 = 4$

$f'(2) = 2 \cdot 2 = 4$

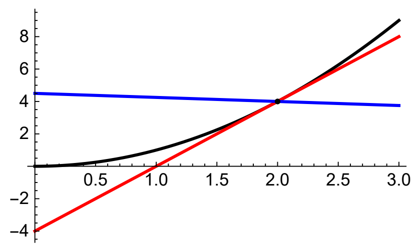
$y = 4 + 4(x - 2)$

Perpendicular line:

$m = -\frac{1}{4}$

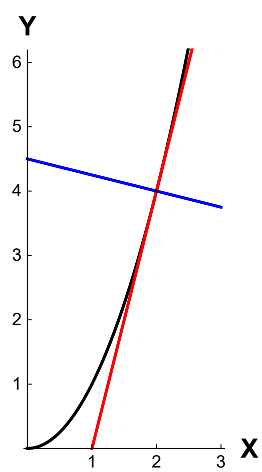
$y = 4 - \frac{1}{4}(x - 4)$

b.



Comments: The graph is correct. But the tangent line, while looking like a tangent line, does not look like it has slope 4. The perpendicular line does not look perpendicular.
The cure is to make sure the scales on both axes are the same.

When doing geometry, we usually want the scales on both axes to be the same. But in other applications we do not care (for example, in a motion problem, why would we want the distance axes with units in meters to have the same scale as the time axes with units in seconds?).



It looks good now.

2.2 Understanding the Derivative

Definition of the Derivative The derivative function $f'(x)$ is

$$\frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx} \approx> f'(x)$$

NOTE The **≈> f '(x)** is considered superfluous.

provided the result of rounding off is the same for every infinitesimal $dx \neq 0$.

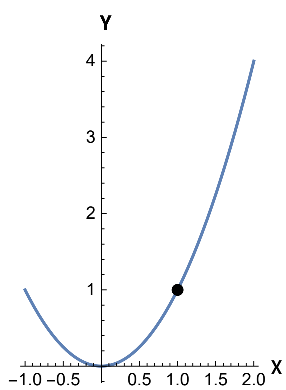
Possible Outcomes Since there are two possible outcomes of rounding off, the same outcomes apply to derivatives.

1. f '(x) is a real number. } extended real number
   f '(x) is $+\infty$ or $-\infty$
2. f '(x) Does Not Exist (DNE)

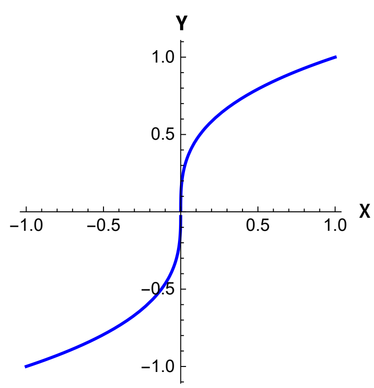
f '(x) is called the ***derivative*** of f(x). f is said to be ***differentiable*** at x if f '(x) is an extended real number because it is meaningful in applications to do so (most mathematicians do not include $\pm\infty$). The process of deriving f '(x) from f(x) is called ***differentiation***.

Examples

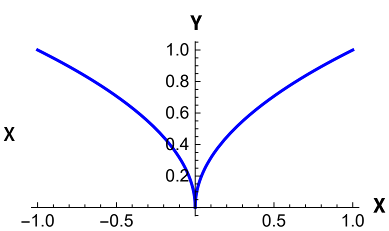
$f(x) = x^2$	$f(x) = x^{1/3}$	$f(x) = x^{2/3}$
$f'(x) = 2x$	$f'(x) = \frac{1}{3x^{2/3}}$	$f'(x) = \frac{2}{3x^{1/3}}$
$f'(1) = 2$, exists	$f'(0) = \frac{1}{0^+} = +\infty$	$f'(0)$ DNE



f '(1) = 2



f '(0) = +∞



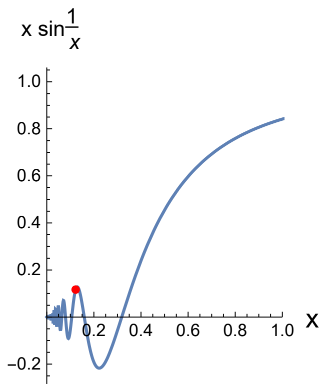
f '(0) DNE

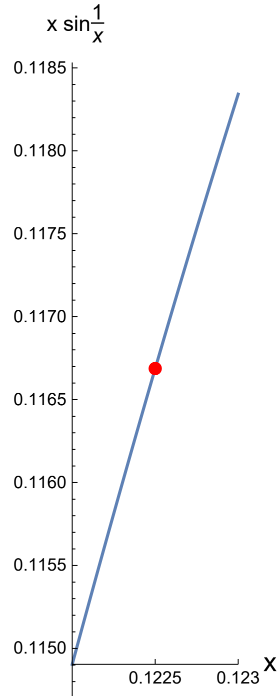
When does a function have a derivative? The following statements are equivalent.

1. f is differentiable at x = a
2. f'(a) exists
3. f is locally linear at x = a
4. f has a tangent line at x = a
5. f is smooth at x = a

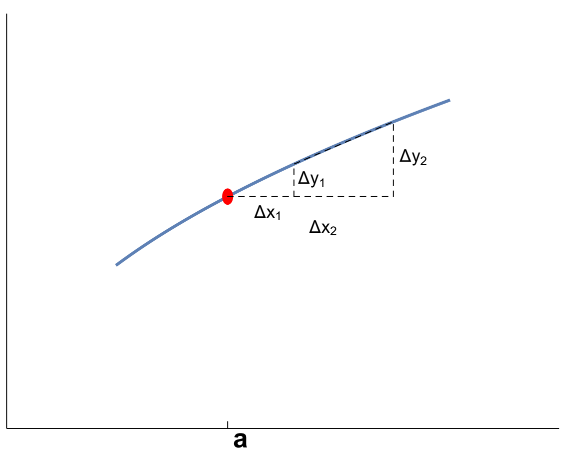
Thanks to *Harvard Consortium Calculus*. ?

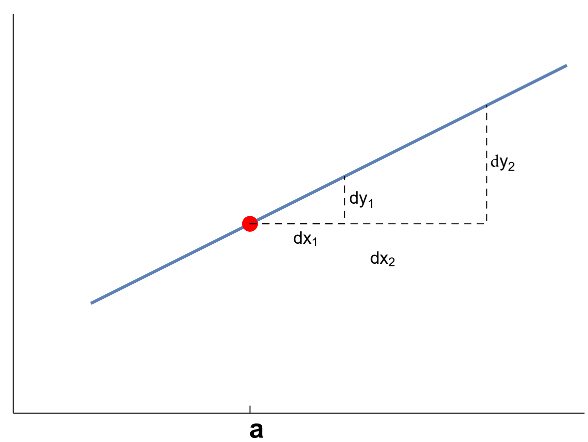
It is a good exercise on occasion to look a point on a curve where it has a derivative and think about the five equivalent properties. Lets look at one in particular, local linearity.





The ‘line’ on the right graph above is a magnification of about 1000 of the left graph near the red dot. So why is local linearity so important for the existence of its derivative?
Look at the graphs below.





The problem in finding the slope at $x = a$ is this. The slope ratios depend on the size of Δx:

$$\frac{\Delta y_1}{\Delta x_1} \neq \frac{\Delta y_2}{\Delta x_2}.$$

Limit people have to go through a complicated of letting Δx approach 0 in order to find

$f'(a)$.

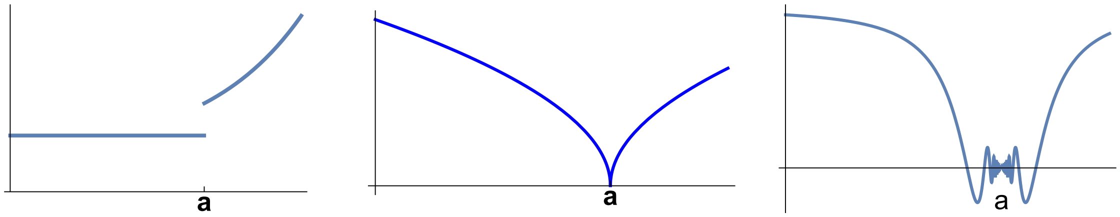
If you take dx to be an infinitesimal, then the slope ratios are still not equal, but:

$$\frac{dy_1}{dx_1} \approx \frac{dy_2}{dx_2},$$

infinitesimally close. So for either calculation, the slope ratio yields

$$\frac{dy}{dx} \approx> f'(a).$$

When does a function not have a derivative? Let us think in terms of smoothness.



f is discontinuous at x = a — **wood planer**

f has a cusp at x = a — **wood scraper**

f highly oscillatory at x = a — **sandpaper**

'Tactile thinking'

Applications

For mathematicians, finding the slope and tangent line to a curve y = f(x) at x = a is a favorite application.

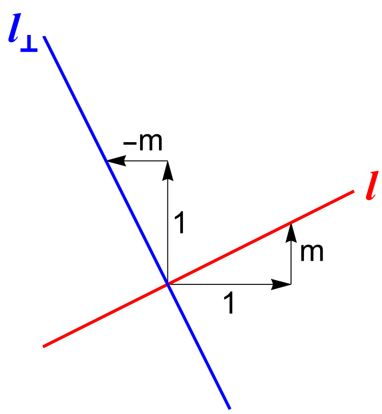
the slope is m = f'(a)

the tangent line is y = f(a) + f'(a)(x - a)

the slope of the perpendicular line is $m_\perp = -\frac{1}{m} = -\frac{1}{f'(a)}$

the perpendicular line is $y = f(a) - \frac{1}{f'(a)}(x - a)$

Note below the relationship between the slope of a line (red) and the slope of a line perpendicular to it (blue).



Slope of the line l is $\frac{\Delta y}{\Delta x} = \frac{m}{1} = m$

Slope of the perpendicular line $l_\perp$ is $m_\perp = \frac{\Delta y}{\Delta x} = \frac{1}{-m} = -\frac{1}{m}$.

For scientists, finding the growth rate (or rate of change) of a quantity is perhaps the most important application.

For example, suppose the mass of a growing melon is $M = t^2$ gm, t in weeks. What is its growth rate when t = 5 weeks?

Answer: Its growth rate then is

$$\frac{dM}{dt} = 2t\Big|_{t=5} = 10\ \frac{\text{grams}}{\text{week}}$$

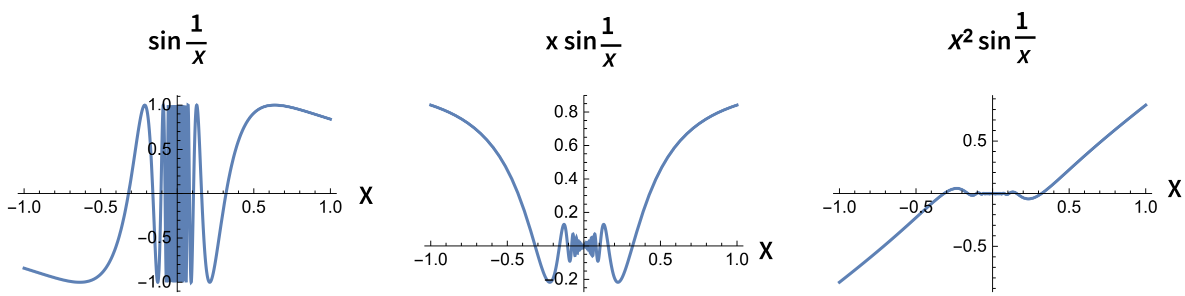
which means during the next week, you expect its mass to increase by about* 10 grams.

****about*** because the curve is not completely straight.

Exercises When the exercise set is small, make sure you spend extra time on the lesson readings!

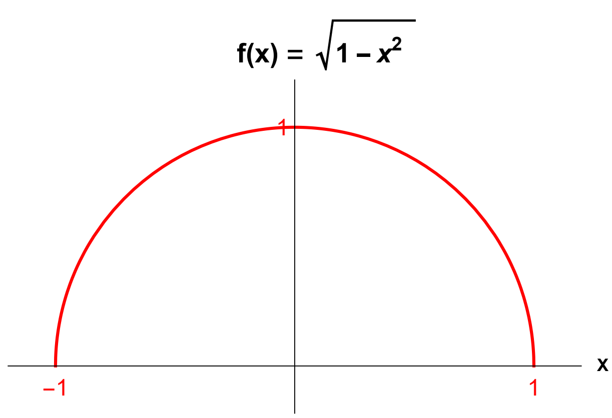
1. Invent graphical examples of your own that illustrate
    1. $f'(1)$ is a real number.
    2. $f'(1)$ is $-\infty$
    3. $f'(1)$ does not exist
2. a. Use a graphing calculator to draw the curve $y = x^3$ for $0 \le x \le 2$.
   Zoom in about $x = 1$ until the curve there looks like a straight line. What was the magnification?

   b. Find the approximate slope at $x = 1$ using a suitable Δx. Compare with the exact answer.
3. Invent graphical examples of your own that illustrate
    a. a discontinuity at $x = 0$.
    b. a cusp at $x = 0$.
    c. highly oscillatory $x = 0$.
4. Consider the graphs below. Which are differentiable at $x = 0$?



5. Use the definition of derivative to find the derivative of $f(x) = \sqrt[3]{x}$ at $x = 0$.

6.



a. $f'(-1^+) =$

b. $f'(1^-) =$

Think graphically. You soon will learn how to find $f'(x)$ analytically.

Note:

$f'(-1^+)$, the '*derivative from the right*' means taking $dx > 0$ at $x = -1$.

$f'(1^-)$, the '*derivative from the left*' means taking $dx < 0$ at $x = 1$.

Solutions

4. No, No, Yes
6. $+\infty$, $-\infty$

2.3 Basic Derivative Rules

You know the Power Rule; $\frac{d}{dx}(x^n) = n\,x^{n-1}$. Next we learn how to differentiate many algebraic combinations of powers of x.

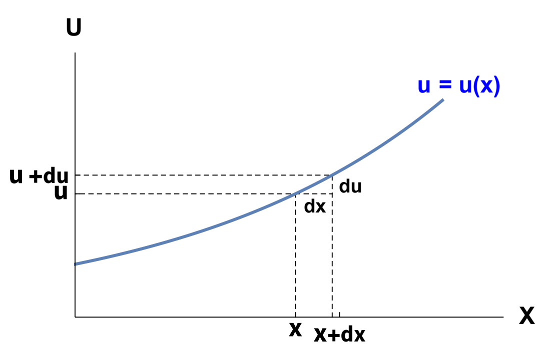
Let u = u(x) and v = v(x) be differentiable.

I. Constant Multiple Rule	$\frac{d}{dx}(c\,u) = c\frac{du}{dx}$
II. Sum Rule	$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$
III. Product Rule	$\frac{d}{dx}(u\,v) = \frac{du}{dx}v + u\frac{dv}{dx}$
IV. Quotient Rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

Note: The Quotient Rule seems to be the hard one to remember. If you memorize the the Product Rule as shown above, the numerator of the Quotient rule is the same as the Product Rule but with a minus sign in front of the second term. That minus sign makes sense because when a denominator increases, the fraction decreases.

Proofs Using Symmetric Applied Function Notation This notation is often used by scientists and engineers because it is useful and intuitive when analyzing problems (and doesn't waste letters).

Let u = u(x) and v = v(x) be differentiable functions



$u(x+dx) = u + du$

Note that there should be no confusion between the dependent variable u and the function u(x).

I. Constant Multiplier Rule $\frac{d}{dx}(c\,u) = c\frac{du}{dx}$

Proof $\frac{d}{dx}(c\,u) = \frac{c\,u(x+dx) - c\,u(x)}{dx}$ definition of derivative

$= \frac{c(u+du) - (c\,u)}{dx}$ applied notation

$= \frac{c\,u + c\,du - c\,u}{dx}$

$= c\frac{du}{dx}$

II. Sum Rule $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

Proof $\frac{d}{dx}(u+v) = \frac{(u(x+dx)+v(x+dx)) - (u(x)+v(x))}{dx}$

$= \frac{((u+du)+(v+dv)) - (u+v)}{dx}$

$= \frac{du+dv}{dx}$

$= \frac{du}{dx} + \frac{dv}{dx}$

III. Product Rule $\frac{d}{dx}(u\,v) = \frac{du}{dx}v + u\frac{dv}{dx}$

Proof $\frac{d}{dx}(u\,v) = \frac{u(x+dx)\,v(x+dx) - u(x)\,v(x)}{dx}$

$= \frac{(u+du)\,(v+dv) - u\,v}{dx}$

$= \frac{(u\,v + u\,dv + du\,v + du\,dv) - u\,v}{dx}$

$= \frac{du}{dx}v + u\frac{dv}{dx} + \frac{du}{dx}dv$

$\approx \frac{du}{dx}v + u\frac{dv}{dx}$

Textbook Proof $(f(x)\,g(x))' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\,g(x+\Delta x) - f(x)\,g(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\,g(x+\Delta x) - f(x+\Delta x)\,g(x) + f(x+\Delta x)\,g(x) - f(x)\,g(x)}{\Delta x}$ Explain this step.

$= \lim_{\Delta x \to 0} f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} + \frac{f(x+\Delta x) - f(x)}{\Delta x}g(x)$

$= f(x)\,g'(x) + f'(x)\,g(x)$

IV. Quotient Rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

Proof $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{u(x+dx)}{v(x+dx)} - \frac{u(x)}{v(x)}}{dx}$

$= \frac{1}{dx}\left(\frac{u+du}{v+dv} - \frac{u}{v}\right)$

$= \frac{1}{dx}\frac{(u+du)\,v - u(v+dv)}{(v+dv)\,v}$

$= \frac{1}{dx}\frac{u\,v + du\,v - u\,v - u\,dv}{(v+dv)\,v}$

$= \frac{1}{dx}\frac{du\,v - u\,dv}{(v+dv)\,v}$

$= \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{(v+dv)\,v}$

$\approx \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

Examples

$\frac{d}{dx}(x^3 - 3x^2 + 5) = 3x^2 - 3 \cdot 2x$

$\frac{d}{dx}((2x+1)(x^3 + 2x^2 + 5)) = 2(x^3 + 2x^2 + 5) + (2x+1)(3x^2 - 2 \cdot 2x)$

$\frac{d}{dx}\left(\frac{x^2+2x+7}{4x-9}\right) = \frac{((2x+2)(4x-9) - (x^2+2x+7)(4)}{(4x-9)^2}$

You can now differentiate all polynomial and rational functions quickly!

How do you differentiate the product of **three factors**? Think of it as two factors.

$\frac{d}{dx}[x(x^2+1)(x^3+2)]$

$= \frac{d}{dx}[x\{(x^2+1)(x^3+2)\}]$

$= 1\{(x^2+1)(x^3+2)\} + x\{2x(x^3+2) + (x^2+1)3x^2\}$

Another way to remember the product rule is to write the sum of uv twice, uv + uv, and then take the derivative of u in the first term and then the derivative of v in the second term. Trying that for uvw we would get $\frac{d}{dx}(uvw) = \frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx}$

Good Derivative Notation Style for y = f(x). Various notations are

$\frac{dy}{dx} \approx f'(x) = Dy = D\,f(x) = D_x y = D_x f(x) = \dot{y} = y'$.

The first is preferred by applied mathematicians. The second by pure mathematicians. The others are for special applications or are out of style. y' is for the poorly motivated (it does not tell you what the independent variable is). The over-dot notation is usually used when the independent variable is t.

Examples

Form	**Preferred Style**
$y = 3x^2 + 2$,	$\frac{dy}{dx} = 6x$
$f(x) = x^5$,	$f'(x) = 5x^4$
$y = f(x)$,	$\frac{dy}{dx} = f'(x)$
$x^3 + 5x$,	$\frac{d}{dx}(x^3 + 5x) = 3x^2 + 5$

Theory Exercises

1T. Use the definition of ≈ **to prove** $\frac{du}{dx}v + u\frac{dv}{dx} + \frac{du}{dx}dv \approx \frac{du}{dx}v + u\frac{dv}{dx}$.

2T. Use the definition of ≈ **to prove** $\frac{\frac{du}{dx}v - u\frac{dv}{dx}}{(v+dv)v} \approx \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$.

3T. Prove $\frac{}{dx}(uvw) = \frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx}$.

Exercises Do not simplify*

1. $y = x^7$
2. $f(x) = x^{5/2}$
3. $\frac{d}{dx}[x^{\pi}]$
4. $\frac{d}{dx}[\pi^{x}]$
5. $\frac{d}{dx}[3x^5]$
6. $\frac{d}{dx}[-7\sqrt{x}]$
7. $\frac{d}{dx}[c\sqrt[3]{x}]$
8. $\frac{d}{dx}\left[\frac{5}{x^3}\right]$
9. $\frac{d}{dx}[3x + 2]$
10. $\frac{d}{dx}[ax^2 + bx - c]$
11. $\frac{d}{dx}[x^3 - 3x^2 - 5x + 2]$
12. $\frac{d}{dx}[x^7 - 3x^6]$
13. $\frac{d}{dx}[2\sqrt{x} - 5\sqrt[3]{x}]$
14. $\frac{d}{dx}[x\sqrt{x} - 1]$
15. $\frac{d}{dt}[t-3]$
16. $\frac{d}{dp}[p^3 + 2p]$
17. $\frac{d}{dx}[(x^2 + 5)(3x^3 - 7x + 5)]$
18. $\frac{d}{dx}[(ax + b)(cx^2 + dx + e)]$
19. $\frac{d}{dx}[(x+1)(x^2 + 2)(x^3 + 3)]$
20. $\frac{d}{dx}[(x+1)(x^2 + 2)(x^3 + 3)(x^4 + 4)]$
21. $\frac{d}{dx}\left[\frac{1}{x+5}\right]$
22. $\frac{d}{dx}\left[\frac{x}{2x+3}\right]$
23. $\frac{d}{dx}\left[\frac{ax+b}{x^2+cx+d}\right]$
24. $\frac{d}{dp}\left[\frac{\sqrt{p}}{p+2}\right]$
25. $\frac{d}{dx}\left[\frac{(x^2+3)(2x^3-7x+4)}{x^2+5}\right]$
26. $\frac{d}{dt}\left[\frac{(t+1)(t+2)}{(t-3)(t-4)(t-5)}\right]$

27. Use the Quotient Rule to prove the Power Rule for n a negative integer. Recall we proved the Power Rule for n a positive integer. Hint: write $\frac{d}{dx}(x^{-n})$.

28. Write the formula for the derivative of **uvwz**.

Solutions

1. $7x^6$
3. $\pi x^{\pi-1}$
5. $15x^4$
7. $\frac{d}{dx}[c\sqrt[3]{x}] = \frac{d}{dx}[cx^{1/3}] = \frac{1}{3}cx^{-2/3}$
9. 3
11. $3x^2 - 6x - 5$
13. $\frac{d}{dx}[2\sqrt{x} - 5\sqrt[3]{x}] = \frac{d}{dx}[2x^{1/2} - x^{1/3}] = x^{-1/2} - \frac{1}{3}x^{-2/3}$
15. 1
17. $2x(3x^3 - 7x + 5) + (x^2 + 5)(9x^2 - 7)$
19. $1(x^2 + 2)(x^3 + 3) + (x + 1)(2x)(x^3 + 3)] + (x + 1)(x^2 + 2)(2x^2)]$
21. $\frac{0-1}{(x+5)^2}$
23. $\frac{a(x^2+cx+d)-(ax+b)(2x+c)}{(x^2+cx+d)^2}$
25. $\frac{[2x(2x^3-7x+4)+(x^2+3)(6x^2-7)]-[(x^2+3)(2x^3-7x+4)](2x)}{(x^2+5)^2}$

* Normally it is not necessary to simplify unless you want to check the 'answer at the back of the book' or you are going to use it for further work or you are compulsive. If you do not simplify, you can see how the answer was calculated when looking at the solutions (depending on the author).

2.4 The Derivatives of Some Transcendental Functions

Today we derive the rest of the formulas for derivatives of functions every beginning calculus student should know.

General Derivative Formulas

I. $\frac{d}{dx}(c\,u) = c\frac{du}{dx}$ $\qquad$ $[c\,f(x)]' = c\,f'(x)$

II. $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$ $\qquad$ $[f(z) + g(x)]' = f'(x) + g'(x)$

III. $\frac{d}{dx}(u\,v) = \frac{du}{dx}v + u\frac{dv}{dx}$ $\qquad$ $[f(x)\,g(x)]' = f'(x)\,g(x) + f(x)\,g'(x)$

IV. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$ $\qquad$ $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}$

Note: remember the numerator of the quotient rule by observing it's the same as the product rule but with a minus sign. The second term is negative because a fraction decreases when its denominator increases.

Special Derivative Formulas

$\frac{d}{dx}(c) = 0$ $\qquad$ $(x^n)' = n\,x^{n-1}$

$\frac{d}{dx}(\sin x) = \cos x$ $\qquad$ $(\cos x)' = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $(\cot x)' = -\csc^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\qquad$ $(\csc x)' = -\csc x \cot x$

$\frac{d}{dx}(e^x) = e^x$ $\qquad$ $(\ln x)' = \frac{1}{x}$

That's all folks! **(for now)**

The bottom row is not universally taught in first semester calculus. However, some disciplines require it. For the rest of us, it gives us more opportunity to practice doing derivatives even for unfamiliar functions. We will only do a quick and dirty introduction to them now. Next semester we will do a careful study of these very important functions.

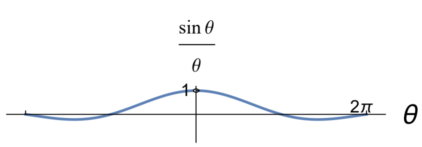
The Trig Functions

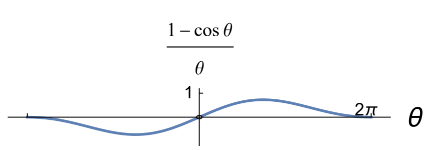
Finding a derivative normally leads to a {0/0} type limit. In section 1.3, we looked at the limits

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$\lim_{h \to 0} \frac{1 - \cos h}{h} = 0$$

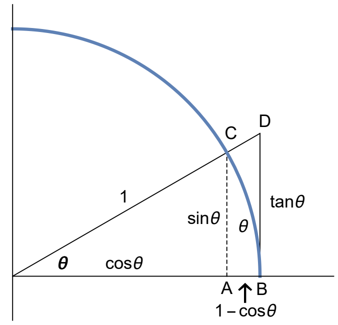
To reinforce these limits look at the graphs below





Hint: Observe values near θ = 0.

or, better yet, see if you can read these limits off of the unit circle.



Observe that as θ approaches 0, sin θ and θ both approach tan θ ⇒

$$\Rightarrow \quad \frac{\sin\theta}{\theta} \approx 1$$

Observe that as θ approaches 0, 1 - cos θ is very small compared to θ

$$\Rightarrow \quad \frac{1-\cos\theta}{\theta} \approx> 0$$

You will also have to recall some identities:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan A = \frac{\sin A}{\cos A}$ $\qquad$ $\cot A = \frac{\cos A}{\sin A}$

$\sec A = \frac{1}{\cos A}$ $\qquad$ $\csc A = \frac{1}{\sin A}$

Proof Let $y = \sin x$

$$\frac{dy}{dx} = \frac{\sin(x+dx) - \sin x}{dx} \qquad \text{definition of derivative}$$

$$= \frac{\sin x \cos dx - \cos x \sin dx - \sin x}{dx}$$

$$= -\frac{1-\cos dx}{dx}\sin x + \frac{\sin dx}{dx}\cos x$$

$$\approx> -0 \cdot \sin x + 1 \cdot \cos x \qquad \text{the trig limits above}$$

$$= \cos x$$

Proofs Let $y = \cos x$. Exercise for you.

Proof Let $y = \tan x$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Proof Let $y = \cot x$. Exercise for you.

Proof Let $y = \sec x$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{0 - 1(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x$$

Proof Let $y = \csc x$. Exercise for you.

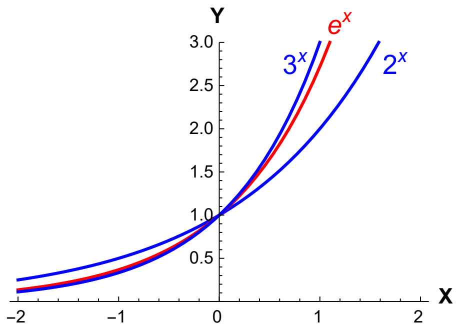
The Natural Exponential and Logarithmic Functions

Exponential functions, $y = b^x$, often are uses to model growth (if $b > 1$) or decay (if $0 < b < 1$).

In advanced applications the base called e is chosen because it has a simple derivative. That base $e = 2.718\cdots$ is called ***Euler's constant***. The ***natural exponential function*** is

$$y = e^x \ .$$

Its graph lies between the graphs of $y = 2^x$ and $y = 3^x$.



If we did more with this topic now, this book would be called

Apex Infinitesimal Calculus,

Early Transcendentals.

Some disciplines require this function in the first semester.

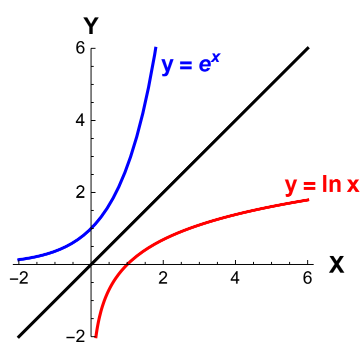
Its derivative formula is derived next semester.

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

The inverse function for $y = e^x$ is obtained by solving for x in terms of y. This cannot be done by elementary algebra. So we give this function a name and let calculators tell us its values. Then

$$x = \ln y \iff y = e^x.$$

ln is pronounced 'natural logarithm' or 'ell-en' or 'lon' (rhymes with Ron). In advanced math ln is often written log. To study this function we interchange x and y and write $y = \ln x$.



The derivative of $y = \ln x \iff x = e^y$ is

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}.$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

You will also need some **properties of the natural logarithmic function**.

1. $\ln(u\,v) = \ln u + \ln v$
2. $\ln \frac{u}{v} = \ln u + \ln v$
3. $\ln u^v = v \ln u$

Note: there are thousands of advanced *named functions*, some of which you may use without a full understanding of them, like e^x and $\ln x$ now. Get used to that!

Exercises 2.4

Terms and Concepts

1. T/F: The Product Rule states that $\frac{d}{dx}\left(x^2 \sin x\right) = 2x\cos x$.

2. T/F: The Quotient Rule states that $\frac{d}{dx}\left(\frac{x^2}{\sin x}\right) = \frac{\cos x}{2x}$.

3. T/F: The derivatives of the trigonometric functions that start with "c" have minus signs in them.

4. What derivative rule is used to extend the Power Rule to include negative integer exponents?

5. T/F: Regardless of the function, there is always exactly one right way of computing its derivative.

6. In your own words, explain what it means to make your answers "clear."

Problems

In Exercises 7 – 10:

(a) Use the Product Rule to differentiate the function.

(b) Manipulate the function algebraically and differentiate without the Product Rule.

(c) Show that the answers from (a) and (b) are equivalent.

7. $f(x) = x(x^2 + 3x)$

8. $g(x) = 2x^2(5x^3)$

9. $h(s) = (2s - 1)(s + 4)$

10. $f(x) = (x^2 + 5)(3 - x^3)$

In Exercises 11 – 14:

(a) Use the Quotient Rule to differentiate the function.

(b)*Manipulate the function algebraically and differentiate without the Quotient Rule.

(c) Show that the answers from (a) and (b) are equivalent.

***Not recommended. Typically yields an ugly looking answer.**

11. $f(x) = \frac{x^2 + 3}{x}$

12. $g(x) = \frac{x^3 - 2x^2}{2x^2}$

13. $h(s) = \frac{3}{4s^3}$

14. $f(t) = \frac{t^2 - 1}{t + 1}$

In Exercises 15 – 36, compute the derivative of the given function.

15. $f(x) = x\sin x$

16. $f(x) = x^2\cos x$

17. $f(x) = e^x \ln x$

18. $f(t) = \frac{1}{t^2}(\csc t - 4)$

19. $g(x) = \frac{x + 7}{x - 5}$

20. $g(t) = \frac{t^5}{\cos t - 2t^2}$

21. $h(x) = \cot x - e^x$

22. $f(x) = \left(\tan x\right)\ln x$

23. $h(t) = 7t^2 + 6t - 2$

24. $f(x) = \frac{x^4 + 2x^3}{x + 2}$

25. $f(x) = \left(3x^2 + 8x + 7\right)e^x$

26. $g(t) = \frac{t^5 - t^3}{e^t}$

27. $f(x) = (16x^3 + 24x^2 + 3x)\frac{7x - 1}{16x^3 + 24x^2 + 3x}$

28. $f(t) = t^5(\sec t + e^t)$

29. $f(x) = \frac{\sin x}{\cos x + 3}$

30. $f(\theta) = \theta^3\sin\theta + \frac{\sin\theta}{\theta^3}$

31. $f(x) = \frac{\cos x}{x} + \frac{x}{\tan x}$

32. $g(x) = e^2\left(\sin(\pi/4) - 1\right)$

33. $g(t) = 4t^3e^t - \sin t\cos t$

34. $h(t) = \frac{t^2\sin t + 3}{t^2\cos t + 2}$

35. $f(x) = x^2e^x\tan x$

36. $g(x) = 2x\sin x\sec x$

In Exercises 37 – 40, find the equations of the tangent and normal lines to the graph of g at the indicated point.

37. $g(s) = e^s(s^2 + 2)$ at $(0, 2)$.

38. $g(t) = t \sin t$ at $(\frac{3\pi}{2}, -\frac{3\pi}{2})$

39. $g(x) = \dfrac{x^2}{x-1}$ at $(2, 4)$

40. $g(\theta) = \dfrac{\cos\theta - 8\theta}{\theta + 1}$ at $(0, 1)$

In Exercises 41 – 44, find the x–values where the graph of the function has a horizontal tangent line.

41. $f(x) = 6x^2 - 18x - 24$

42. $f(x) = x \sin x$ on $[-1, 1]$

43. $f(x) = \dfrac{x}{x+1}$

44. $f(x) = \dfrac{x^2}{x+1}$

In Exercises 45 – 48, find the requested derivative.

45. $f(x) = x \sin x$; find $f''(x)$.

46. $f(x) = x \sin x$; find $f^{(4)}(x)$.

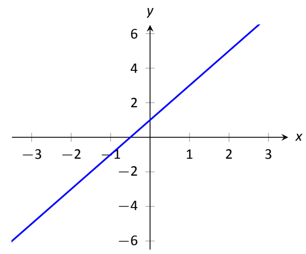
47. $f(x) = \csc x$; find $f''(x)$.

48. $f(x) = (x^3 - 5x + 2)(x^2 + x - 7)$; find $f^{(8)}(x)$.

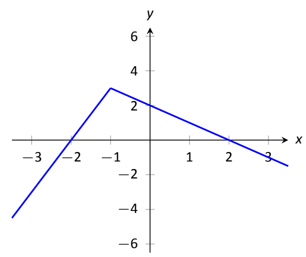
Review

In Exercises 49 – 50, use the graph of $f(x)$ to sketch $f'(x)$.

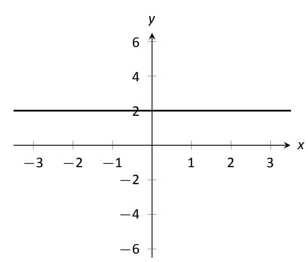
49.

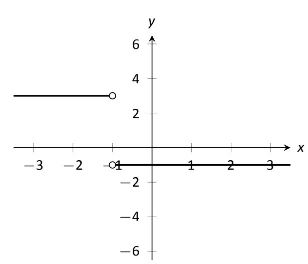


50.



Solutions 2.4

1. F
2. F
3. T
4. Quotient Rule
5. F
6. Answers will vary.
7. (a) $f'(x) = (x^2 + 3x) + x(2x + 3)$
   (b) $f'(x) = 3x^2 + 6x$
   (c) They are equal.
8. (a) $g'(x) = 4x(5x^3) + 2x^2(15x^2)$
   (b) $g'(x) = 50x^4$
   (c) They are equal.
9. (a) $h'(s) = 2(s + 4) + (2s - 1)(1)$
   (b) $h'(s) = 4s + 7$
   (c) They are equal.
10. (a) $f'(x) = 2x(3 - x^3) + (x^2 + 5)(-3x^2)$
    (b) $f'(x) = -5x^4 - 15x^2 + 6x$
    (c) They are equal.
11. (a) $f'(x) = \frac{x(2x)-(x^2+3)1}{x^2}$
    (b) $f'(x) = 1 - \frac{3}{x^2}$
    (c) They are equal.
12. (a) $g'(x) = \frac{2x^2(3x^2-4x)-(x^3-2x^2)(4x)}{4x^4}$
    (b) $g'(x) = 1/2$
    (c) They are equal.
13. (a) $h'(s) = \frac{4s^3(0)-3(12s^2)}{16s^6}$
    (b) $h'(s) = -9/4s^{-4}$
    (c) They are equal.
14. (a) $f'(t) = \frac{(t+1)(2t)-(t^2-1)(1)}{(t+1)^2}$
    (b) $f(t) = t - 1$ when $t \neq -1$, so $f'(t) = 1$.
    (c) They are equal.
15. $f'(x) = \sin x + x \cos x$
16. $f'(x) = 2x \cos x - x^2 \sin x$
17. $f'(x) = e^x \ln x + e^x \frac{1}{x}$
18. $f'(t) = \frac{-2}{t^3}(\csc t - 4) + \frac{1}{t^2}(-\csc t \cot t)$
19. $g'(x) = \frac{-12}{(x-5)^2}$
20. $g'(t) = \frac{(\cos t-2t^2)(5t^4)-(t^5)(-\sin t-4t)}{(\cos t-2t^2)^2}$
21. $h'(x) = -\csc^2 x - e^x$
22. $f'(x) = \big(\sec^2 x\big) \ln x + \big(\tan x\big) \frac{1}{x}$
23. $h'(t) = 14t + 6$
24. (a) $f'(x) = \frac{(x+2)(4x^3+6x^2)-(x^4+2x^3)(1)}{(x+2)^2}$
    (b) $f(x) = x^3$ when $x \neq -2$, so $f'(x) = 3x^2$.
    (c) They are equal.
25. $f'(x) = \big(6x + 8\big)e^x + \big(3x^2 + 8x + 7\big)e^x$
26. $g'(t) = \frac{e^t(5t^4-3t^2)-(t^5-t^3)e^t}{(e^t)^2}$
27. $f'(x) = 7$
28. $f'(t) = 5t^4(\sec t + e^t) + t^5(\sec t \tan t + e^t)$
29. $f'(x) = \frac{\sin^2(x)+\cos^2(x)+3\cos(x)}{(\cos(x)+3)^2}$
30. $f'(\theta) = 3\theta^2 \sin\theta + \theta^3 \cos\theta + \frac{\theta^3\cos\theta-(\sin\theta)(3\theta^2)}{\theta^6}$
31. $f'(x) = \frac{-x\sin x-\cos x}{x^2} + \frac{\tan x-x\sec^2 x}{\tan^2 x}$
32. $g'(x) = 0$
33. $g'(t) = 12t^2e^t + 4t^3e^t - \cos^2 t + \sin^2 t$
34. $f'(x) = \frac{(t^2\cos t+2)(2t\sin t+t^2\cos t)-(t^2\sin t+3)(2t\cos t-t^2\sin t)}{(t^2\cos t+2)^2}$
35. $f'(x) = 2xe^x \tan x = x^2e^x \tan x + x^2e^x \sec^2 x$
36. $g'(x) = 2\sin x \sec x + 2x\cos x \sec x + 2x \sin x \sec x \tan x =$
    $2\tan x + 2x + 2x\tan^2 x = 2\tan x + 2x\sec^2 x$
37. Tangent line: $y = 2x + 2$
    Normal line: $y = -1/2x + 2$
38. Tangent line: $y = -(x - \frac{3\pi}{2}) - \frac{3\pi}{2} = -x$
    Normal line: $y = (x - \frac{3\pi}{2}) - \frac{3\pi}{2} = x - 3\pi$
39. Tangent line: $y = 4$
    Normal line: $x = 2$
40. Tangent line: $y = -9x + 1$
    Normal line: $y = 1/9x + 1$
41. $x = 3/2$
42. $x = 0$
43. $f'(x)$ is never 0.
44. $x = -2, 0$
45. $f''(x) = 2\cos x - x \sin x$
46. $f^{(4)}(x) = -4\cos x + x\sin x$
47. $f''(x) = \cot^2 x \csc x + \csc^3 x$
48. $f^{(8)} = 0$
49. 



2.5 The Chain Rule

In applications you rarely meet a simple function like cos x. Its more likely to look like cos(2.34x + 7.29) or cos(2πft), the composition of the cosine function with another function.

Suppose y = f(g(x)). Taking it apart: y = f(u), u = g(x)

V. The Chain Rule $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Proof The chain rule proves itself by hyperreal algebra. One possible problem. By the definition of derivative, in $\frac{du}{dx}$, dx cannot be 0. But du could be. Then the first factor $\frac{dy}{du}$ would have an illegal denominator, 0. The cure: disallow, as required by the definition of derivative, the dx that yields du = 0.

Example $y = (2x + 4)^3$. Think $y = u^3$, u = 2x + 4. Then

$$\frac{d}{dx}((2x + 4)^3 = 3(2x + 4)^2 \cdot 2$$

Often with y = f(g(x)) we think of the chain rule as the derivative of a composite function as the derivative of the outside function times the derivative function. Some prefer the pure math version

[f(g(x))]' = f '(g(x))·g'(x) thinking (outside fn)' · (inside fn)'

Whence the name '***Chain Rule***'?

Suppose y = f(g(h(x))) $\Longleftrightarrow$ y = f(u), u = g(v), v = h(x)?

Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx},$$

the terms being connected in a chainlike way.

Example

$$\frac{d}{dx}(\cos^3(x^2 + 5)); \quad y = u^3, u = \cos v, v = x^2 + 5$$

$$= 3 \cdot \cos^2(x^2 + 5) \cdot (-\sin(x^2 + 5)) \cdot 2x$$

Special Derivative Formulas in Chain rule form

It is a good idea to memorize these formulas in this form.

Let u = u(x) and v = v(x) be differentiable functions. Then

$\frac{d}{dx}(c) = 0$ | $\frac{d}{dx}(u^n) = n\,u^{n-1}\frac{du}{dx}$

$\frac{d}{dx}(\sin u) = \cos u\,\frac{du}{dx}$ | $\frac{d}{dx}(\cos u) = -\sin u\,\frac{du}{dx}$

$\frac{d}{dx}(\tan u) = \sec^2 u\,\frac{du}{dx}$ | $\frac{d}{dx}(\cot u) = -\csc^2 u\,\frac{du}{dx}$

$\frac{d}{dx}(\sec u) = \sec u \tan u\,\frac{du}{dx}$ | $\frac{d}{dx}(\csc u) = -\csc u \cot u\,\frac{du}{dx}$

$\frac{d}{dx}(e^u) = e^u\,\frac{du}{dx}$ | $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx}(a^u) = a^u \ln a\,\frac{du}{dx}$ | $\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a}\frac{du}{dx}$

Readings: The Chain Rule

We have covered almost all of the derivative rules that deal with combinations of two (or more) functions. The operations of addition, subtraction, multiplication (including by a constant) and division led to the Sum and Difference rules, the Constant Multiple Rule, the Power Rule, the Product Rule and the Quotient Rule. To complete the list of differentiation rules, we look at the last way two (or more) functions can be combined: the process of composition (i.e. one function "inside" another).

One example of a composition of functions is $f(x) = \cos(x^2)$. We currently do not know how to compute this derivative. If forced to guess, one would likely guess $f'(x) = -\sin(2x)$, where we recognize $-\sin x$ as the derivative of $\cos x$ and $2x$ as the derivative of x^2. However, this is not the case; $f'(x) \neq -\sin(2x)$. In Example 2.5.4 we'll see the correct answer, which employs the new rule this section introduces, the **Chain Rule**.

Theorem The Chain Rule

Let g be a differentiable function on an interval I, let the range of g be a subset of the interval J, and let f be a differentiable function on J. Then $y = f(g(x))$ i s a differentiable function on I, and

$$y' = f'(g(x)) \cdot g'(x).$$

Example 2.5.1 Using the Chain Rule

Find the derivatives of the following functions:

1. $y = \sin 2x$ 2. $y = \ln(4x^3 - 2x^2)$ 3. $y = e^{-x^2}$

Solution

1. Consider $y = \sin 2x$. Recognize that this is a composition of functions, where $f(x) = \sin x$ and $g(x) = 2x$. Thus

$$y' = f'(g(x)) \cdot g'(x) = \cos(2x) \cdot 2 = 2\cos 2x.$$

2. Recognize that $y = \ln(4x^3 - 2x^2)$ is the composition of $f(x) = \ln x$ and $g(x) = 4x^3 - 2x^2$. Also, recall that

$$\frac{d}{dx}\Big(\ln x\Big) = \frac{1}{x}.$$

This leads us to:

$$y' = \frac{1}{4x^3 - 2x^2} \cdot (12x^2 - 4x) = \frac{12x^2 - 4x}{4x^3 - 2x^2} = \frac{4x(3x-1)}{2x(2x^2 - x)} = \frac{2(3x-1)}{2x^2 - x}.$$

3. Recognize that $y = e^{-x^2}$ is the composition of $f(x) = e^x$ and $g(x) = -x^2$. Remembering that $f'(x) = e^x$, we have

$$y' = e^{-x^2} \cdot (-2x) = (-2x)e^{-x^2}.$$

Example 2.5.2 Using the Chain Rule to find a tangent line

Let $f(x) = \cos x^2$. Find the equation of the line tangent to the graph of f at $x = 1$.

Solution The tangent line goes through the point $(1, f(1)) \approx (1, 0.54)$ with slope $f'(1)$. To find f', we need the Chain Rule.

$f'(x) = -\sin(x^2) \cdot (2x) = -2x \sin x^2$. Evaluated at $x = 1$, we have $f'(1) = -2\sin 1 \approx -1.68$. Thus the equation of the tangent line is

$$y = -1.68(x-1) + 0.54.$$

The tangent line is sketched along with f in Figure 2.5.1.

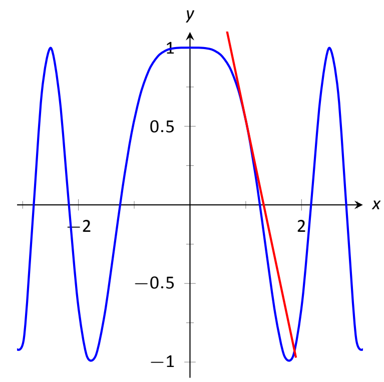


Figure 2.5.1: $f(x) = \cos x^2$ sketched along with its tangent line at $x = 1$.

The Chain Rule is used often in taking derivatives. Because of this, one can become familiar with the basic process and learn patterns that facilitate finding derivatives quickly. For instance,

$$\frac{d}{dx}\Big(\ln(\text{anything})\Big) = \frac{1}{\text{anything}} \cdot (\text{anything})' = \frac{(\text{anything})'}{\text{anything}}.$$

Example 2.5.3 Using the Chain Rule multiple times

Find the derivative of $y = \tan^5(6x^3 - 7x)$.

Solution Recognize that we have the $g(x) = \tan(6x^3 - 7x)$ function "inside" the $f(x) = x^5$ function; that is, we have $y = \big(\tan(6x^3 - 7x)\big)^5$. We begin using the Generalized Power Rule; in this first step, we do not fully compute the derivative. Rather, we are approaching this step–by–step.

$$y' = 5\big(\tan(6x^3 - 7x)\big)^4 \cdot g'(x).$$

We now find $g'(x)$. We again need the Chain Rule;

$$g'(x) = \sec^2(6x^3 - 7x) \cdot (18x^2 - 7).$$

Combine this with what we found above to give

$$\begin{aligned} y' &= 5\big(\tan(6x^3 - 7x)\big)^4 \cdot \sec^2(6x^3 - 7x) \cdot (18x^2 - 7) \\ &= (90x^2 - 35)\sec^2(6x^3 - 7x)\tan^4(6x^3 - 7x). \end{aligned}$$

This function is frankly a ridiculous function, possessing no real practical value. It is very difficult to graph, as the tangent function has many vertical asymptotes and $6x^3 - 7x$ grows so very fast. The important thing to learn from this is that the derivative can be found. In fact, it is not "hard;" one can take several simple steps and should be careful to keep track of how to apply each of these steps.

Exercises 2.5

Terms and Concepts

1. T/F: The Chain Rule describes how to evaluate the derivative of a composition of functions.

2. T/F: The Generalized Power Rule states that $\frac{d}{dx}\Big(g(x)^n\Big) = n\big(g(x)\big)^{n-1}$.

3. T/F: $\frac{d}{dx}\big(\ln(x^2)\big) = \frac{1}{x^2}$.

4. T/F: $\frac{d}{dx}\big(3^x\big) \doteq 1.1 \cdot 3^x$.

5. T/F: $\frac{dx}{dy} = \frac{dx}{dt} \cdot \frac{dt}{dy}$

6. $f(x) = \big(\ln x + x^2\big)^3$

Problems

In Exercises 7 – 36, compute the derivative of the given function.

7. $f(x) = (4x^3 - x)^{10}$

8. $f(t) = (3t - 2)^5$

9. $g(\theta) = (\sin\theta + \cos\theta)^3$

10. $h(t) = e^{3t^2+t-1}$

11. $f(x) = \big(\ln x + x^2\big)^3$

12. $f(x) = 2^{x^3+3x}$

13. $f(x) = \big(x + \frac{1}{x}\big)^4$

14. $f(x) = \cos(3x)$

15. $g(x) = \tan(5x)$

16. $h(\theta) = \tan\big(\theta^2 + 4\theta\big)$

17. $g(t) = \sin\big(t^5 + \frac{1}{t}\big)$

18. $h(t) = \sin^4(2t)$

19. $p(t) = \cos^3(t^2 + 3t + 1)$

20. $f(x) = \ln(\cos x)$

21. $f(x) = \ln(x^2)$

22. $f(x) = 2\ln(x)$

23. $g(r) = 4^r$

24. $g(t) = 5^{\cos t}$

25. $g(t) = 15^2$

26. $m(w) = \frac{3^w}{2^w}$

27. $h(t) = \frac{2^t + 3}{3^t + 2}$

28. $m(w) = \frac{3^w + 1}{2^w}$

29. $f(x) = \frac{3^{x^2} + x}{2^{x^2}}$

30. $f(x) = x^2\sin(5x)$

31. $f(x) = (x^2 + x)^5(3x^4 + 2x)^3$

32. $g(t) = \cos(t^2 + 3t)\sin(5t - 7)$

33. $f(x) = \sin(3x + 4)\cos(5 - 2x)$

34. $g(t) = \cos(\frac{1}{t})e^{5t^2}$

35. $f(x) = \frac{\sin\big(4x + 1\big)}{(5x - 9)^3}$

36. $f(x) = \frac{(4x + 1)^2}{\tan(5x)}$

In Exercises 37 – 40, find the equations of tangent and normal lines to the graph of the function at the given point. Note: the functions here are the same as in Exercises 7 through 10.

37. $f(x) = (4x^3 - x)^{10}$ at $x = 0$

38. $f(t) = (3t - 2)^5$ at $t = 1$

39. $g(\theta) = (\sin\theta + \cos\theta)^3$ at $\theta = \pi/2$

40. $h(t) = e^{3t^2+t-1}$ at $t = -1$

41. Compute $\frac{d}{dx}\big(\ln(kx)\big)$ two ways:

    (a) Using the Chain Rule, and

    (b) by first using the logarithm rule $\ln(ab) = \ln a + \ln b$, then taking the derivative.

42. Derivative of sin x, x in degrees.

43. Find the ***second derivative chain rule***.

Solutions 2.5

1. T, 2. F, 3. F, 4. T, 5. T

6. f'(x) = $3\left(\ln x + x^2\right)^2\left(\frac{1}{x} + 2x\right)$
7. $f'(x) = 10(4x^3 - x)^9 \cdot (12x^2 - 1) = (120x^2 - 10)(4x^3 - x)^9$
8. $f'(t) = 15(3t - 2)^4$
9. $g'(\theta) = 3(\sin\theta + \cos\theta)^2(\cos\theta - \sin\theta)$
10. $h'(t) = (6t + 1)e^{3t^2+t-1}$
11. $f'(x) = 3\left(\ln x + x^2\right)2\left(\frac{1}{x} + 2x\right)$
12. $f'(x) = (\ln 2)(2^{x^3+3x})(3x^2 + 3)$
13. $f'(x) = 4\left(x + \frac{1}{x}\right)^3\left(1 - \frac{1}{x^2}\right)$
14. $f'(x) = -3\sin(3x)$
15. $g'(x) = 5\sec^2(5x)$
16. $h'(\theta) = \sec^2\left(\theta^2 + 4\theta\right)(2\theta + 4)$
17. $g'(t) = \cos\left(t^5 + \frac{1}{t}\right)\left(5t^4 - \frac{1}{t^3}\right)$
18. $h'(t) = 8\sin^3(2t)\cos(2t)$
19. $p'(t) = -3\cos^2(t^2 + 3t + 1)\sin(t^2 + 3t + 1)(2t + 3)$
20. $f'(x) = -\tan x$
21. $f'(x) = 2/x$
22. $f'(x) = 2/x$
23. $g'(r) = \ln 4 \cdot 4^r$
24. $g'(t) = -\ln 5 \cdot 5^{\cos t}\sin t$
25. $g'(t) = 0$
26. $m'(w) = \ln(3/2)(3/2)^w$
27. $f'(x) = \frac{(3^t+2)\left((\ln 2)2^t\right) - (2^t+3)\left((\ln 3)3^t\right)}{(3^t+2)^2}$
28. $m'(w) = \frac{2^w\left(\ln 3 \cdot 3^w - \ln 2 \cdot (3^w+1)\right)}{2^{2w}}$
29. $f'(x) = \frac{2^{x^2}(\ln 3 \cdot 3^{x^2}2x+1) - (3^{x^2}+x)(\ln 2 \cdot 2^{x^2}2x)}{2^{2x^2}}$
30. $f'(x) = 5x^2\cos(5x) + 2x\sin(5x)$
31. $f'(x) =$
$5(x^2+x)^4(2x+1)(3x^4+2x)^3 + 3(x^2+x)^5(3x^4+2x)^2(12x^3+2)$
32. $g'(t) = 5\cos(t^2+3t)\cos(5t-7) - (2t+3)\sin(t^2+3t)\sin(5t-7)$
33. $f'(x) = 3\cos(3x + 4)\cos(5 - 2x) + 2\sin(3x + 4)\sin(5 - 2x)$
34. $g'(t) = 10t\cos\left(\frac{1}{t}\right)e^{5t^2} + \frac{1}{t^2}\sin\left(\frac{1}{t}\right)e^{5t^2}$
35. $f'(x) = \frac{4(5x-9)^3\cos(4x+1) - 15\sin(4x+1)(5x-9)^2}{(5x-9)^6}$
36. $f'(x) = \frac{8\tan(5x)(4x+1) - 5(4x+1)^2\sec^2(5x)}{\tan^2(5x)}$
37. Tangent line: $y = 0$
    Normal line: $x = 0$
38. Tangent line: $y = 15(t - 1) + 1$
    Normal line: $y = -1/15(t - 1) + 1$
39. Tangent line: $y = -3(\theta - \pi/2) + 1$
    Normal line: $y = 1/3(\theta - \pi/2) + 1$
40. Tangent line: $y = -5e(t + 1) + e$
    Normal line: $y = 1/(5e)(t + 1) + e$
41. In both cases the derivative is the same: $1/x$.
42. Hint: convert x to radians.
43.

$y = f(g(x)) \Leftrightarrow y = f(u),\ u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$ Chain Rule

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\right)\frac{du}{dx} + \frac{dy}{dx}\frac{d^2u}{dx^2}$$ Product Rule

$$\frac{d^2y}{dx^2} = \frac{du}{dx}\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx} + \frac{dy}{dx}\frac{d^2u}{dx^2}$$ Hyperreal algebra

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{dx}\frac{d^2u}{dx^2}$$

2.6 Implicit Differentiation

In the previous sections we learned to find the derivative, $\frac{dy}{dx}$, or y', when y is given *explicitly* as a function of x. That is, if we know $y = f(x)$ for some function f, we can find y'. For example, given $y = 3x^2 - 7$, we can easily find $y' = 6x$. (Here we explicitly state how x and y are related. Knowing x, we can directly find y.)

Sometimes the relationship between y and x is not explicit; rather, it is *implicit*. For instance, we might know that $x^2 - y = 4$. This equality defines a relationship between x and y; if we know x, we could figure out y. Can we still find y'? In this case, sure; we solve for y to get $y = x^2 - 4$ (hence we now know y explicitly) and then differentiate to get $y' = 2x$.

Sometimes the *implicit* relationship between x and y is complicated. Suppose we are given $\sin(y) + y^3 = 6 - x^3$. A graph of this implicit function is given in Figure 2.6.1. In this case there is absolutely no way to solve for y in terms of elementary functions. The surprising thing is, however, that we can still find y' via a process known as **implicit differentiation**.

Implicit differentiation is a technique based on the Chain Rule that is used to find a derivative when the relationship between the variables is given implicitly rather than explicitly (solved for one variable in terms of the other).

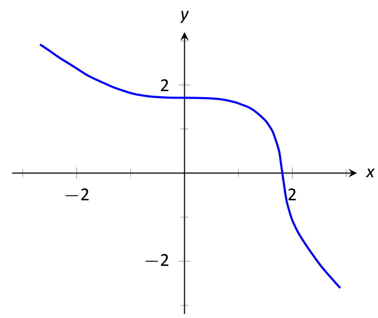


Figure 2.6.1: A graph of the implicit function $\sin(y) + y^3 = 6 - x^3$.

We begin by reviewing the Chain Rule. Let f and g be functions of x. Then

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x).$$

Compact Theory

$y = f(g(x)) \iff y = f(u),\ u = g(x)$

$\frac{dy}{dx} \approx \frac{dy}{du}\frac{du}{dx}$

by hyperreal algebra.

Suppose now that $y = g(x)$. We can rewrite the above as

$$\frac{d}{dx}\Big(f(y)\Big) = f'(y) \cdot y', \quad \text{or} \quad \frac{d}{dx}\Big(f(y)\Big) = f'(y) \cdot \frac{dy}{dx}. \tag{2.1}$$

These equations look strange; the key concept to learn here is that we can find y' even if we don't exactly know how y and x relate.

We demonstrate this process in the following example.

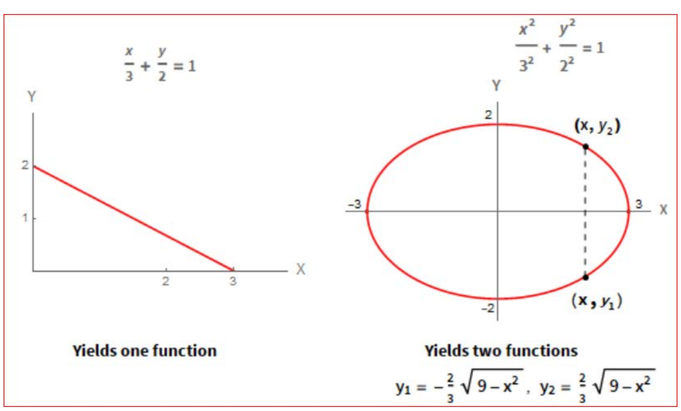
Example 2.6.1 Using Implicit Differentiation

Find y' given that $\sin(y) + y^3 = 6 - x^3$.

Solution We start by taking the derivative of both sides (thus maintaining the equality.) We have :

$$\frac{d}{dx}\Big(\sin(y) + y^3\Big) = \frac{d}{dx}\Big(6 - x^3\Big).$$

Example



The right hand side is easy; it returns $-3x^2$.

The left hand side requires more consideration. We take the derivative term–by–term. Using the technique derived from Equation 2.1 above, we can see that

$$\frac{d}{dx}\Big(\sin y\Big) = \cos y \cdot y'.$$

We apply the same process to the y^3 term.

$$\frac{d}{dx}\Big(y^3\Big) = \frac{d}{dx}\Big((y)^3\Big) = 3(y)^2 \cdot y'.$$

Putting this together with the right hand side, we have

$$\cos(y)y' + 3y^2y' = -3x^2.$$

Now solve for y'.

$$\begin{aligned} \cos(y)y' + 3y^2y' &= -3x^2. \\ \big(\cos y + 3y^2\big)y' &= -3x^2 \\ y' &= \frac{-3x^2}{\cos y + 3y^2} \end{aligned}$$

This equation for y' probably seems unusual for it contains both x and y terms. How is it to be used? We'll address that next.

Implicit functions are generally harder to deal with than explicit functions. With an explicit function, given an x value, we have an explicit formula for computing the corresponding y value. With an implicit function, one often has to find x and y values *at the same time* that satisfy the equation. It is much easier to demonstrate that a given point satisfies the equation than to actually find such a point.

For instance, we can affirm easily that the point $(\sqrt[3]{6}, 0)$ lies on the graph of the implicit function $\sin y + y^3 = 6 - x^3$. Plugging in 0 for y, we see the left hand side is 0. Setting $x = \sqrt[3]{6}$, we see the right hand side is also 0; the equation is satisfied. The following example finds the equation of the tangent line to this function at this point.

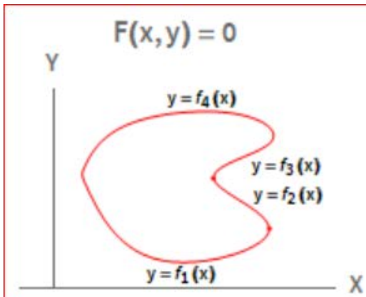
Example 2.6.2 Using Implicit Differentiation to find a tangent line

Find the equation of the line tangent to the curve of the implicitly defined function $\sin y + y^3 = 6 - x^3$ at the point $(\sqrt[3]{6}, 0)$.

SOLUTION In Example 2.6.1 we found that

$$y' = \frac{-3x^2}{\cos y + 3y^2}.$$

Example



F(x, y) = 0 yields four functions.

We find the slope of the tangent line at the point $(\sqrt[3]{6}, 0)$ by substituting $\sqrt[3]{6}$ for x and 0 for y. Thus at the point $(\sqrt[3]{6}, 0)$, we have the slope as

$$y' = \frac{-3(\sqrt[3]{6})^2}{\cos 0 + 3 \cdot 0^2} = \frac{-3\sqrt[3]{36}}{1} \doteq -9.91.$$

Therefore the equation of the tangent line to the implicitly defined function $\sin y + y^3 = 6 - x^3$ at the point $(\sqrt[3]{6}, 0)$ is

$$y = -3\sqrt[3]{36}(x - \sqrt[3]{6}) + 0 \doteq -9.91x + 18.$$

The curve and this tangent line are shown in Figure 2.6.2.

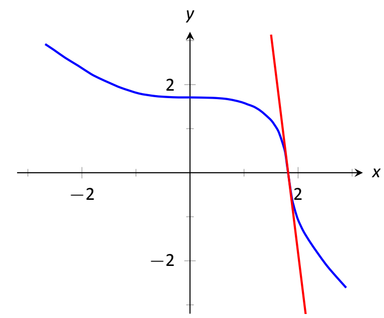


Figure 2.6.2: The function $\sin y + y^3 = 6 - x^3$ and its tangent line at the point $(\sqrt[3]{6}, 0)$.

This suggests a general method for implicit differentiation. For the steps below assume y is a function of x.

1. Take the derivative of each term in the equation. Treat the x terms like normal. When taking the derivatives of y terms, the usual rules apply except that, because of the Chain Rule, we need to multiply each term by y'.
2. Get all the y' terms on one side of the equal sign and put the remaining terms on the other side.
3. Factor out y'; solve for y' by dividing.

Practical Note: When working by hand, it may be beneficial to use the symbol $\frac{dy}{dx}$ instead of y', as the latter can be easily confused for y or y^1.

Example 2.6.3 Using Implicit Differentiation

Given the implicitly defined function $y^3 + x^2y^4 = 1 + 2x$, find y'.

Solution We will take the implicit derivatives term by term. The derivative of y^3 is $3y^2y'$.

The second term, x^2y^4, is a little tricky. It requires the Product Rule as it is the product of two functions of x: x^2 and y^4. Its derivative is $x^2(4y^3y') + 2xy^4$. The first part of this expression requires a y' because we are taking the derivative of a y term. The second part does not require it because we are taking the derivative of x^2.

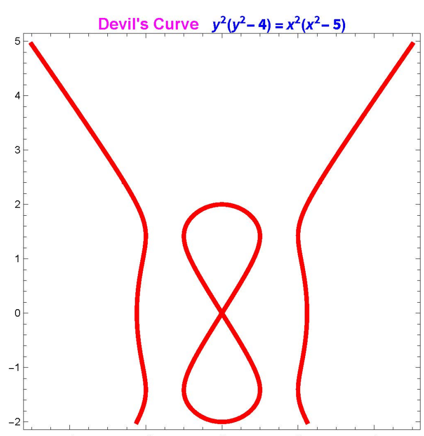
The derivative of the right hand side is easily found to be 2. In all, we get:

$$3y^2y' + 4x^2y^3y' + 2xy^4 = 2.$$

Move terms around so that the left side consists only of the y' terms and the right side consists of all the other terms:

$$3y^2y' + 4x^2y^3y' = 2 - 2xy^4.$$

Example



FYVE*

For Your Viewing Enjoyment

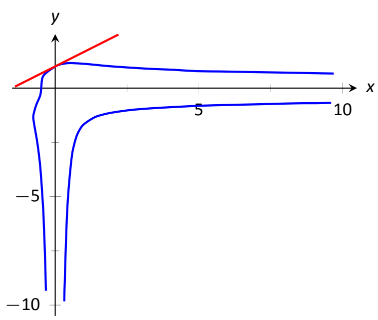


Figure 2.6.3: A graph of the implicitly defined function $y^3 + x^2y^4 = 1 + 2x$ along with its tangent line at the point $(0, 1)$.

Factor out y' from the left side and solve to get

$$y' = \frac{2 - 2xy^4}{3y^2 + 4x^2y^3}.$$

To confirm the validity of our work, let's find the equation of a tangent line to this function at a point. It is easy to confirm that the point $(0, 1)$ lies on the graph of this function. At this point, $y' = 2/3$. So the equation of the tangent line is $y = 2/3(x - 0) + 1$. The function and its tangent line are graphed in Figure 2.6.3.

Notice how our function looks much different than other functions we have seen. For one, it fails the vertical line test. Such functions are important in many areas of mathematics, so developing tools to deal with them is also important.

Example 2.6.4 Using Implicit Differentiation

Given the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$, find y'.

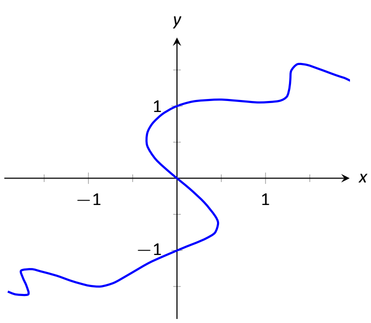


Figure 2.6.4: A graph of the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$.

Solution Differentiating term by term, we find the most difficulty in the first term. It requires both the Chain and Product Rules.

$$\begin{aligned}\frac{d}{dx}\Big(\sin(x^2y^2)\Big) &= \cos(x^2y^2)\cdot\frac{d}{dx}\Big(x^2y^2\Big)\\ &= \cos(x^2y^2)\cdot\big(x^2(2yy') + 2xy^2\big)\\ &= 2(x^2yy' + xy^2)\cos(x^2y^2).\end{aligned}$$

We leave the derivatives of the other terms to the reader. After taking the derivatives of both sides, we have

$$2(x^2yy' + xy^2)\cos(x^2y^2) + 3y^2y' = 1 + y'.$$

We now have to be careful to properly solve for y', particularly because of the product on the left. It is best to multiply out the product. Doing this, we get

$$2x^2y\cos(x^2y^2)y' + 2xy^2\cos(x^2y^2) + 3y^2y' = 1 + y'.$$

From here we can safely move around terms to get the following:

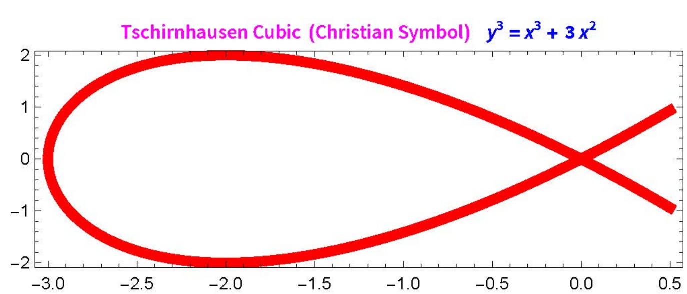
$$2x^2y\cos(x^2y^2)y' + 3y^2y' - y' = 1 - 2xy^2\cos(x^2y^2).$$

Then we can solve for y' to get

$$y' = \frac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2) + 3y^2 - 1}.$$

Example

FYVE*



A graph of this implicit function is given in Figure 2.6.4. It is easy to verify that the points $(0,0)$, $(0,1)$ and $(0,-1)$ all lie on the graph. We can find the slopes of the tangent lines at each of these points using our formula for y'.

At $(0,0)$, the slope is -1.

At $(0,1)$, the slope is $1/2$.

At $(0,-1)$, the slope is also $1/2$.

The tangent lines have been added to the graph of the function in Figure 2.6.5.

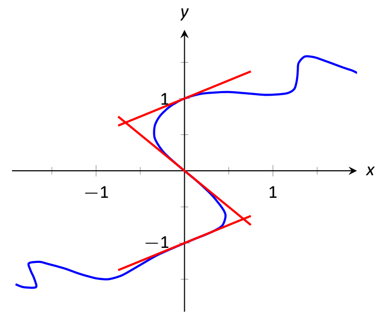


Figure 2.6.5: A graph of the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$ and certain tangent lines.

Quite a few "famous" curves have equations that are given implicitly. We can use implicit differentiation to find the slope at various points on those curves. We investigate two such curves in the next examples.

Example 2.6.5 Finding slopes of tangent lines to a circle

Find the slope of the tangent line to the circle $x^2+y^2 = 1$ at the point $(1/2, \sqrt{3}/2)$.

Solution Taking derivatives, we get $2x+2yy' = 0$. Solving for y' gives:

$$y' = \frac{-x}{y}.$$

This is a clever formula. Recall that the slope of the line through the origin and the point (x, y) on the circle will be y/x. We have found that the slope of the tangent line to the circle at that point is the opposite reciprocal of y/x, namely, $-x/y$. Hence these two lines are always perpendicular.

At the point $(1/2, \sqrt{3}/2)$, we have the tangent line's slope as

$$y' = \frac{-1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}} \doteq -0.577.$$

A graph of the circle and its tangent line at $(1/2, \sqrt{3}/2)$ is given in Figure 2.6.6, along with a thin dashed line from the origin that is perpendicular to the tangent line. (It turns out that all normal lines to a circle pass through the center of the circle.)

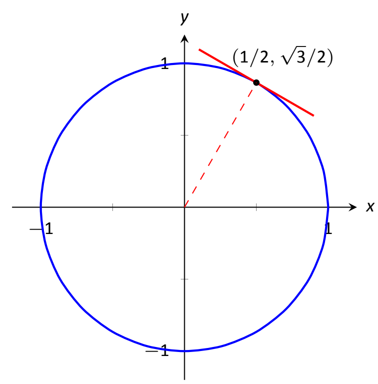


Figure 2.6.6: The unit circle with its tangent line at $(1/2, \sqrt{3}/2)$.

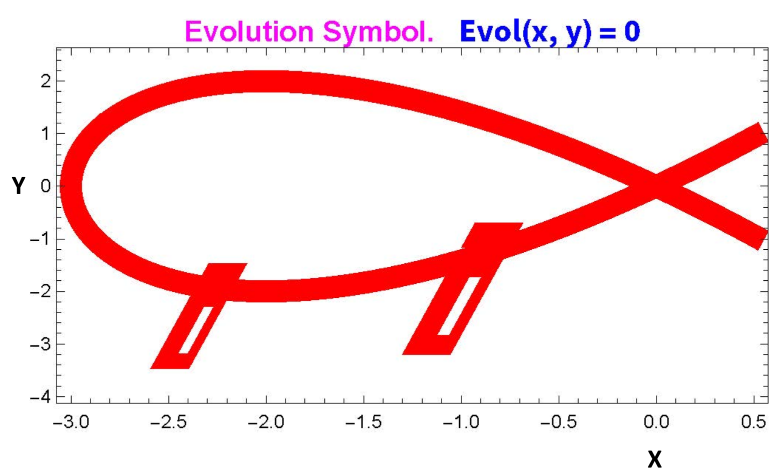
This section has shown how to find the derivatives of implicitly defined functions, whose graphs include a wide variety of interesting and unusual shapes. Implicit differentiation can also be used to further our understanding of "regular" differentiation.

One hole in our current understanding of derivatives is this: what is the derivative of the square root function? That is,

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right) = ?$$

Example

FYVE



We allude to a possible solution, as we can write the square root function as a power function with a rational (or, fractional) power. We are then tempted to apply the Power Rule and obtain

$$\frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

The trouble with this is that the Power Rule was initially defined only for positive integer powers, $n > 0$. While we did not justify this at the time, generally the Power Rule is proved using something called the Binomial Theorem, which deals only with positive integers. The Quotient Rule allowed us to extend the Power Rule to negative integer powers. Implicit Differentiation allows us to extend the Power Rule to rational powers, as shown below.

Let $y = x^{m/n}$, where m and n are integers with no common factors (so $m = 2$ and $n = 5$ is fine, but $m = 2$ and $n = 4$ is not). We can rewrite this explicit function implicitly as $y^n = x^m$. Now apply implicit differentiation.

$$\begin{aligned}
y &= x^{m/n} \\
y^n &= x^m \\
\frac{d}{dx}\left(y^n\right) &= \frac{d}{dx}\left(x^m\right) \\
n \cdot y^{n-1} \cdot y' &= m \cdot x^{m-1} \\
y' &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \quad \text{(now substitute } x^{m/n} \text{ for } y\text{)} \\
&= \frac{m}{n}\frac{x^{m-1}}{\left(x^{m/n}\right)^{n-1}} \quad \text{(apply lots of algebra)} \\
&= \frac{m}{n}x^{(m-n)/n} \\
&= \frac{m}{n}x^{m/n-1}.
\end{aligned}$$

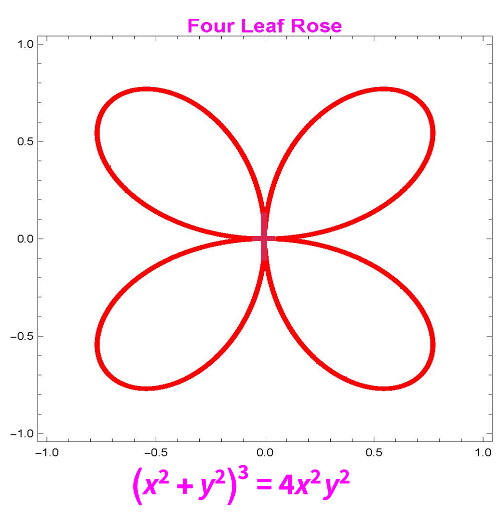
The above derivation is the key to the proof extending the Power Rule to rational powers. Using limits, we can extend this once more to include *all* powers, including irrational (even transcendental!) powers, giving the following theorem.

Theorem 2.6.1 Power Rule for Differentiation

Let $f(x) = x^n$, where $n \neq 0$ is a real number. Then f is differentiable on its domain, except possibly at $x = 0$, and $f'(x) = n \cdot x^{n-1}$.

Example

FYVE



This theorem allows us to say the derivative of x^{π} is $\pi x^{\pi-1}$.

We now apply this final version of the Power Rule in the next example, the second investigation of a "famous" curve.

Example 2.6.6 Using the Power Rule

Find the slope of $x^{2/3} + y^{2/3} = 8$ at the point $(8, 8)$.

Solution This is a particularly interesting curve called an *astroid*. It is the shape traced out by a point on the edge of a circle that is rolling around inside of a larger circle, as shown in Figure 2.6.7.

To find the slope of the astroid at the point $(8, 8)$, we take the derivative implicitly.

$$\begin{aligned}\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' &= 0\\ \frac{2}{3}y^{-1/3}y' &= -\frac{2}{3}x^{-1/3}\\ y' &= -\frac{x^{-1/3}}{y^{-1/3}}\\ y' &= -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}.\end{aligned}$$

Plugging in $x = 8$ and $y = 8$, we get a slope of -1. The astroid, with its tangent line at $(8, 8)$, is shown in Figure 2.6.8.

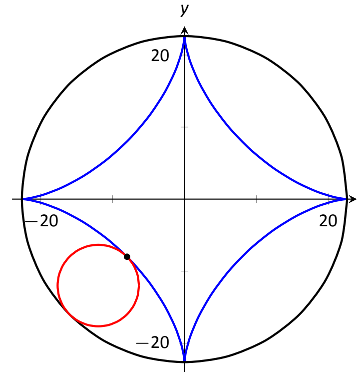


Figure 2.6.7: An astroid, traced out by a point on the smaller circle as it rolls inside the larger circle.

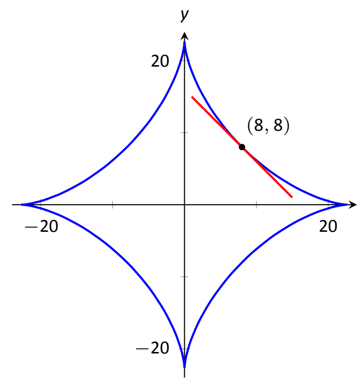


Figure 2.6.8: An astroid with a tangent line.

Implicit Differentiation and the Second Derivative

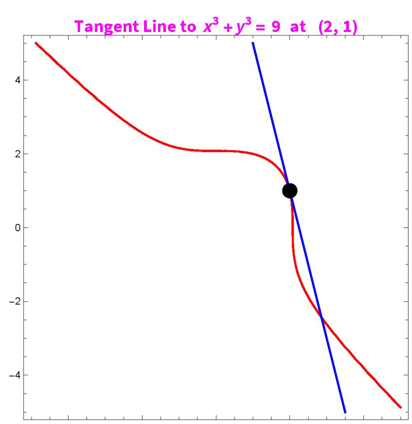
We can use implicit differentiation to find higher order derivatives. In theory, this is simple: first find $\frac{dy}{dx}$, then take its derivative with respect to x. In practice, it is not hard, but it often requires a bit of algebra. We demonstrate this in an example.

Example 2.6.7 Finding the second derivative

Given $x^2 + y^2 = 1$, find $\dfrac{d^2y}{dx^2} = y''$.

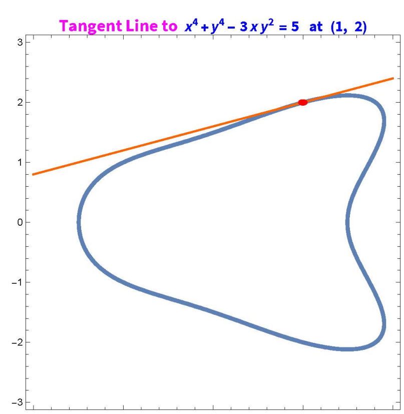
Solution We found that $y' = \frac{dy}{dx} = -x/y$ in Example 2.6.5. To find y'',

Example



FYVE

Example



Logarithmic Differentiation Preview There is still one type of function we cannot handle.

$$y = f(x)^{g(x)}$$

The solution is to use Property 3 of logarithms:

$$\ln(A^B) = B\ln(A)$$

Logarithmic differentiation is an important tool in areas of applications.
For now we content ourselves with an interesting mathematics example.

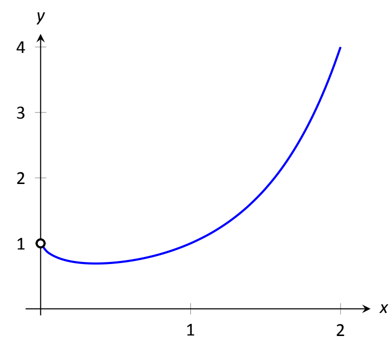


Figure 2.6.9: A plot of $y = x^x$.

Example 2.6.8 Using Logarithmic Differentiation
Given $y = x^x$, use logarithmic differentiation to find y'.

Solution As suggested above, we start by taking the natural log of both sides then applying implicit differentiation.

$$\begin{aligned}
y &= x^x \\
\ln(y) &= \ln(x^x) && \text{(apply logarithm rule)} \\
\ln(y) &= x\ln x && \text{(now use implicit differentiation)} \\
\frac{d}{dx}\Big(\ln(y)\Big) &= \frac{d}{dx}\Big(x\ln x\Big) \\
\frac{y'}{y} &= \ln x + x\cdot\frac{1}{x} \\
\frac{y'}{y} &= \ln x + 1 \\
y' &= y\big(\ln x + 1\big) && \text{(substitute } y = x^x\text{)} \\
y' &= x^x\big(\ln x + 1\big).
\end{aligned}$$

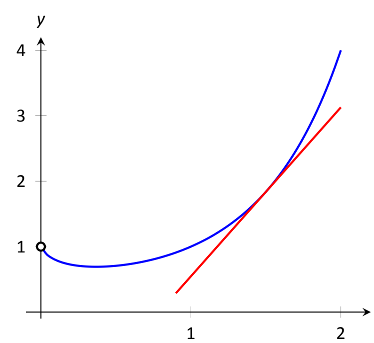


Figure 2.6.10: A graph of $y = x^x$ and its tangent line at $x = 1.5$.

To "test" our answer, let's use it to find the equation of the tangent line at $x = 1.5$. The point on the graph our tangent line must pass through is $(1.5, 1.5^{1.5})$ $(1.5, 1.837)$. Using the equation for y', we find the slope as

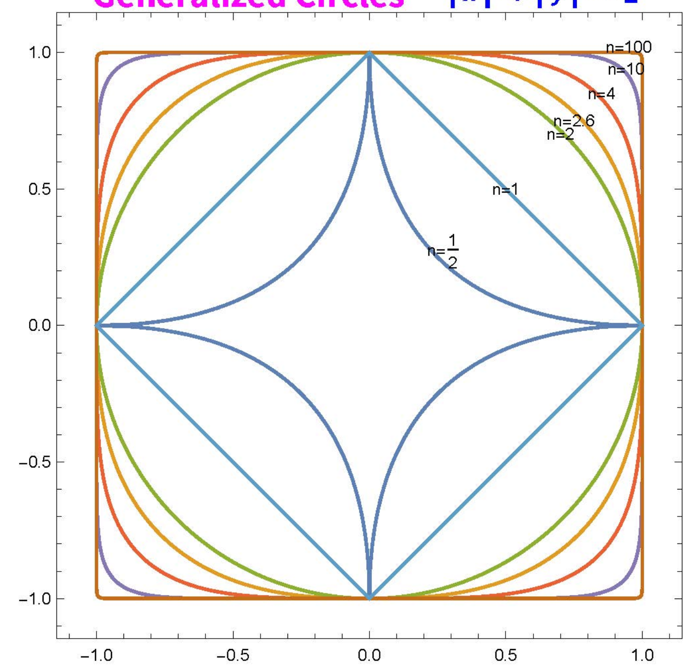
$$y' = 1.5^{1.5}\big(\ln 1.5 + 1\big) \doteq 1.837(1.405) \doteq 2.582.$$

Thus the equation of the tangent line is $y = 1.6833(x - 1.5) + 1.837$. Figure 2.6.10 graphs $y = x^x$ along with this tangent line.

Implicit differentiation proves to be useful as it allows us to find the instantaneous rates of change of a variety of functions. In particular, it extended the Power Rule to rational exponents, which we then extended to all real numbers. In the next calculus course, implicit differentiation will be used to find the derivatives of *inverse* functions, such as $y = \sin^{-1} x$.

Example

Generalized Circles $|x|^n + |y|^n = 1$



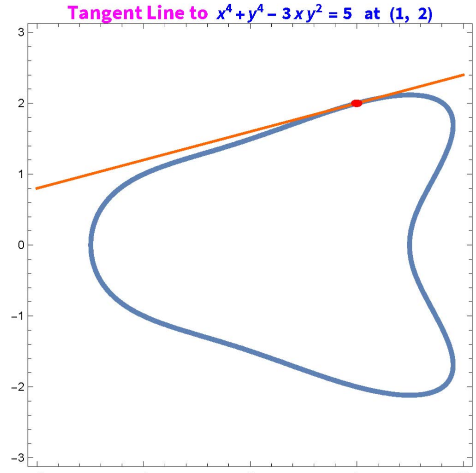
Scientific American
circa 1975

For table tops :

$n = 1/2$	intimate but dangerous corners
$n = 1$	square, still dangerous
$n = 2$	circle, where to sit ?
$n = 2.6$	perfect (Scandinavian designer)
$n \geq 3$	too square.

Example
FYVE

Tangent Line to $x^4 + y^4 - 3xy^2 = 5$ at $(1, 2)$



Exercises 2.6

Terms and Concepts

1. In your own words, explain the difference between implicit functions and explicit functions.

2. Implicit differentiation is based on what other differentiation rule?

3. T/F: Implicit differentiation can be used to find the derivative of $y = \sqrt{x}$.

4. T/F: Implicit differentiation can be used to find the derivative of $y = x^{3/4}$.

Problems

In Exercises 5 – 12, compute the derivative of the given function.

5. $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

6. $f(x) = \sqrt[3]{x} + x^{2/3}$

7. $f(t) = \sqrt{1-t^2}$

8. $g(t) = \sqrt{t}\sin t$

9. $h(x) = x^{1.5}$

10. $f(x) = x^{\pi} + x^{1.9} + \pi^{1.9}$

11. $g(x) = \dfrac{x+7}{\sqrt{x}}$

12. $f(t) = \sqrt[5]{t}(\sec t + e^t)$

In Exercises 13 – 25, find $\dfrac{dy}{dx}$ using implicit differentiation.

13. $x^4 + y^2 + y = 7$

14. $x^{2/5} + y^{2/5} = 1$

15. $\cos(x) + \sin(y) = 1$

16. $\dfrac{x}{y} = 10$

17. $\dfrac{y}{x} = 10$

18. $x^2e^2 + 2^y = 5$

19. $x^2\tan y = 50$

20. $(3x^2 + 2y^3)^4 = 2$

21. $(y^2 + 2y - x)^2 = 200$

22. $\dfrac{x^2+y}{x+y^2} = 17$

23. $\dfrac{\sin(x)+y}{\cos(y)+x} = 1$

24. $\ln(x^2 + y^2) = e$

25. $\ln(x^2 + xy + y^2) = 1$

26. Show that $\dfrac{dy}{dx}$ is the same for each of the following implicitly defined functions.

    (a) $xy = 1$

    (b) $x^2y^2 = 1$

    (c) $\sin(xy) = 1$

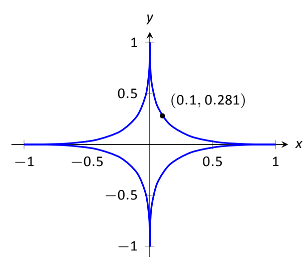
    (d) $\ln(xy) = 1$

In Exercises 27 – 32, find the equation of the tangent line to the graph of the implicitly defined function at the indicated points. As a visual aid, each function is graphed.

27. $x^{2/5} + y^{2/5} = 1$

    (a) At $(1, 0)$.

    (b) At $(0.1, 0.281)$ (which does not *exactly* lie on the curve, but is very close).

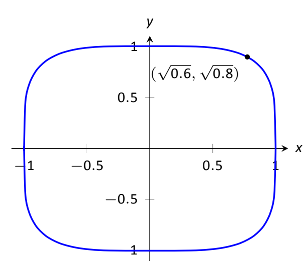


28. $x^4 + y^4 = 1$

    (a) At $(1, 0)$.

    (b) At $(\sqrt{0.6}, \sqrt{0.8})$.

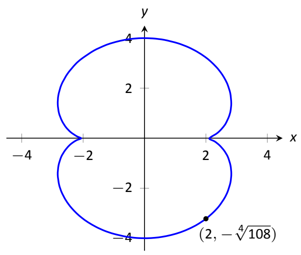
    (c) At $(0, 1)$.



29. $(x^2 + y^2 - 4)^3 = 108y^2$

   (a) At $(0, 4)$.

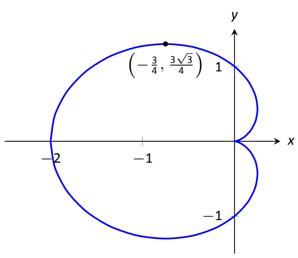
   (b) At $(2, -\sqrt[4]{108})$.



30. $(x^2 + y^2 + x)^2 = x^2 + y^2$

   (a) At $(0, 1)$.

   (b) At $\left(-\frac{3}{4}, \frac{3\sqrt{3}}{4}\right)$.

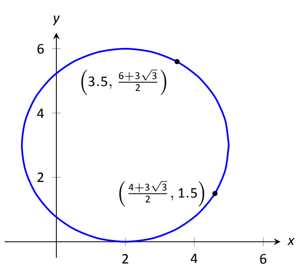


31. $(x - 2)^2 + (y - 3)^2 = 9$

   (a) At $\left(\frac{7}{2}, \frac{6 + 3\sqrt{3}}{2}\right)$.

Example

   (b) At $\left(\frac{4 + 3\sqrt{3}}{2}, \frac{3}{2}\right)$.

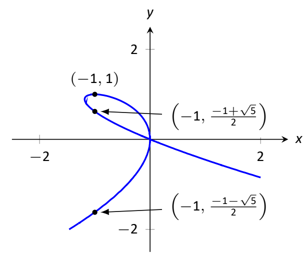


32. $x^2 + y^3 + 2xy = 0$

   (a) At $(-1, 1)$.

   (b) At $\left(-1, \frac{1}{2}(-1 + \sqrt{5})\right)$.

   (c) At $\left(-1, \frac{1}{2}(-1 - \sqrt{5})\right)$.



In Exercises 33 – 36, an implicitly defined function is given. Find $\frac{d^2y}{dx^2}$. Note: these are the same problems used in Exercises 13 through 16.

33. $x^4 + y^2 + y = 7$

34. $x^{2/5} + y^{2/5} = 1$

35. $\cos x + \sin y = 1$

36. $\frac{x}{y} = 10$

Solution 2.6

1. Answers will vary.
2. The Chain Rule.
3. T
4. T
5. $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$
6. $f'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-1/3} = \frac{1}{3\sqrt[3]{x^2}} + \frac{2}{3\sqrt[3]{x}}$
7. $f'(t) = \frac{-t}{\sqrt{1-t^2}}$
8. $g'(t) = \sqrt{t}\cos t + \frac{\sin t}{2\sqrt{t}}$
9. $h'(x) = 1.5x^{0.5} = 1.5\sqrt{x}$
10. $f'(x) = \pi x^{\pi-1} + 1.9x^{0.9}$
11. $g'(x) = \frac{\sqrt{x}(1)-(x+7)(1/2x^{-1/2})}{x} = \frac{1}{2\sqrt{x}} - \frac{7}{2\sqrt{x^3}}$
12. $f'(t) = \frac{1}{5}x^{-4/5}(\sec t + e^t) + \sqrt[5]{t}(\sec t\tan t + e^t)$
13. $\frac{dy}{dx} = \frac{-4x^3}{2y+1}$
14. $\frac{dy}{dx} = -\frac{y^{3/5}}{x^{3/5}}$
15. $\frac{dy}{dx} = \sin(x)\sec(y)$
16. $\frac{dy}{dx} = \frac{y}{x}$
17. $\frac{dy}{dx} = \frac{y}{x}$
18. $\frac{dy}{dx} = -\frac{e^x x(x+2)2^{-y}}{\ln|2|}$
19. $\frac{dy}{dx} = -\frac{2\sin(y)\cos(y)}{x}$
20. $\frac{dy}{dx} = -\frac{x}{y^2}$
21. $\frac{dy}{dx} = \frac{1}{2y+2}$
22. If one takes the derivative of the equation, as shown, using the Quotient Rule, one finds $\frac{dy}{dx} = \frac{x^2+2xy^2-y}{2x^2y-x+y^2}$.
    If one first clears the denominator and writes $x^2 + y = 17(x+y^2)$ then takes the derivative of both sides, one finds $\frac{dy}{dx} = \frac{2x-17}{34y-1}$.
    These expressions, by themselves, are not equal. However, for values of x and y that satisfy the original equation (i.e, for x and y such that $\frac{x^2+y}{x+y^2} = 17$), these expressions are equal.
23. If one takes the derivative of the equation, as shown, using the Quotient Rule, one finds $\frac{dy}{dx} = \frac{-\cos(x)(x+\cos(y))+\sin(x)+y}{\sin(y)(\sin(x)+y)+x+\cos(y)}$.
    If one first clears the denominator and writes $\sin(x) + y = \cos(y) + x$ then takes the derivative of both sides, one finds $\frac{dy}{dx} = \frac{1-\cos(x)}{1+\sin(y)}$.
    These expressions, by themselves, are not equal. However, for values of x and y that satisfy the original equation (i.e, for x and y such that $\frac{\sin(x)+y)}{\cos(y)+x)} = 1$), these expressions are equal.
24. $\frac{dy}{dx} = -\frac{x}{y}$
25. $\frac{dy}{dx} = -\frac{2x+y}{2y+x}$
26. In each, $\frac{dy}{dx} = -\frac{y}{x}$.
27. (a) $y = 0$
    (b) $y = -1.859(x-0.1) + 0.281$
28. (a) $x = 1$
    (b) $y = -\frac{3\sqrt{3}}{8}(x - \sqrt{.6}) + \sqrt{.8} \approx -0.65(x-0.775) + 0.894$
    (c) $y = 1$
29. (a) $y = 4$
    (b) $y = 0.93(x-2) + \sqrt[4]{108}$
30. (a) $y = -1/3x + 1$
    (b) $y = 3\sqrt{3}/4$
31. (a) $y = -\frac{1}{\sqrt{3}}(x - \frac{7}{2}) + \frac{6+3\sqrt{3}}{2}$
    (b) $y = \sqrt{3}(x - \frac{4+3\sqrt{3}}{2}) + \frac{3}{2}$
32. (a) $y = 1$
    (b) $y = -\frac{2}{\sqrt{5}}(x+1) + \frac{1}{2}(-1+\sqrt{5})$
    (c) $y = \frac{2}{\sqrt{5}}(x+1) + \frac{1}{2}(-1-\sqrt{5})$
33. $\frac{d^2y}{dx^2} = \frac{(2y+1)(-12x^2)+4x^3\left(2\frac{-4x^3}{2y+1}\right)}{(2y+1)^2}$
34. $\frac{d^2y}{dx^2} = \frac{3}{5}\frac{y^{3/5}}{x^{8/5}} + \frac{3}{5}\frac{1}{yx^{6/5}}$
35. $\frac{d^2y}{dx^2} = \frac{\cos x\cos y+\sin^2 x\tan y}{\cos^2 y}$
36. $\frac{d^2y}{dx^2} = 0$

3: The Graphical Behavior of Functions

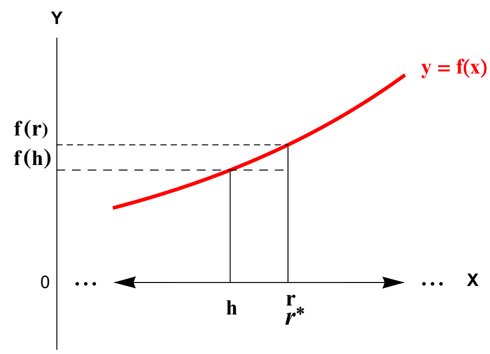
3.1 The Extreme Value Theorem

This section presents some very important properties shared by continuous functions. Their hyperreal proofs are easy (Their real number based proofs are quite difficult and are normally omitted in a beginning calculus course). While these theorems may seem obvious and unexciting, they form a basis for further important results in the calculus and as an expert you will want to see their proofs and understand their importance.

The main result we need now is the ***Extreme Value Theorem***. The other theorems in this section should be a part of your general background knowledge of continuous functions.

An important preliminary note

A pertinent observation follows. In the proof we will calculate the value of a continuous function f at a finite hyperreal number h infinitesimally close to a real number r*; then h ≈> r and since f is a continuous function, f(h) ≈> f(r).



Another important preliminary note

Closed Sequence Principle Every mathematical question which can be answered for a finite sequence $r_1, r_2, \cdots, r_n$ can be answered for a closed (infinite terminating) sequence $h_1, h_2, \cdots, h_N$.

For example

2^{-4} is the least element of the finite sequence $\{1, 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}\}$.

2^{N_o} is the greatest element of the infinite sequence $\{1, 2^1, 2^2, \cdots, 2^{N_o}\}$.

But this cannot always be done for a non–terminating sequence:

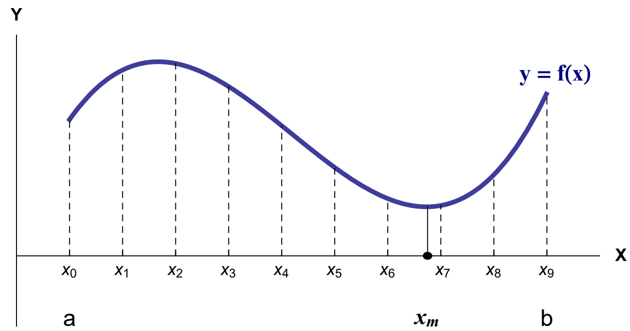
$\{1, 2^{-1}, 2^{-2}, 2^{-3}, \cdots\}$ does not have a least element!

The Extreme Value Theorem

Finding the maximum and minimum value of a function is important in many applications. For example, a manufacturer normally wants to maximize the income function I(x) or minimize the cost function C(x) for manufacturing x items. The following theorem gives an important case where the maximum and minimum values are guaranteed to exist. Calculus will then be very useful in finding these values.

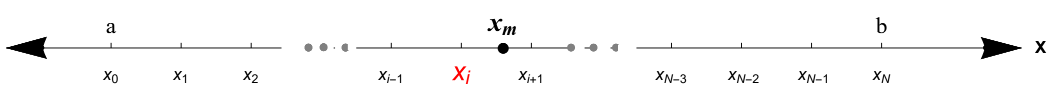
(This important theorem is stated in ***Apex***, Section 3.1, without proof.)

Extreme Value Theorem Let f be continuous on the closed interval $a \le x \le b$. Then f has a maximum and a minimum on the interval.



First let us look at how the proof works approximately using real numbers to find the approximate minimum value of f shown above on the interval $a \le x \le b$. Subdivide the interval into 9 equal finite parts by the numbers $x_0, x_1, x_2, \cdots, x_9$. Then compute the sequence of values $f(x_0), f(x_1), f(x_2), \cdots, f(x_9)$. The minimum value of the sequence is $f(x_7)$. So the minimum value may occur at $x_m \doteq x_7$ and the minimum value may be about $f(x_m) \doteq f(x_7)$. One problem is, of course, that the solution is only approximate; in fact, this solution may be extremely bad because the function could behave very badly between the calculated values.

Proof Subdivide the interval $a \le x \le b$ into an infinite number N of subdivisions of infinitesimal length $\frac{b-a}{N}$ by the sequence $x_0, x_1, x_2, \cdots, x_N$.

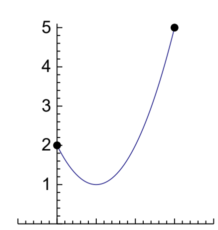


Then compute the sequence of values $\{f(x_0), f(x_1), f(x_2), \cdots, f(x_N)\}$. Suppose the minimum value of this sequence occurs at x_i. By the continuity of f, $x_i \approx> x_m$, a real number. Then $f(x_i) \approx> f(x_m)$, a real number which is the minimum value of f on the interval.

End of Proof

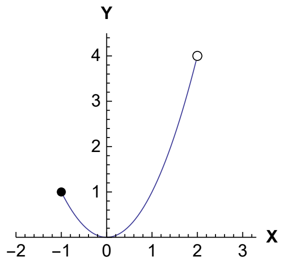
The proof of the existence of a maximum at is similar and is left as an exercise.
Note that this theorem is so difficult to prove using ϵ-δ methods that it is omitted from many textbooks.

Example Find the extreme values of $f(x) = 1 + (x - 1)^2$ on the interval $0 \le x \le 3$.



Since f is continuous on the interval, the extreme values are guaranteed to exist. From the graph, the minimum value $y = 1$ occurs at the vertex of the parabola, $x = 1$. The maximum value $y = 5$ occurs at the endpoint $x = 3$. (Without the theorem which guarantees the existence of a maximum, you might forget to look at end-points or not be sure that the extreme values actually exist.)

Example Find the extreme values of $f(x) = x^2$ on the interval $-1 \leq x < 2$.



f is continuous, but the interval is not closed. So there are no guarantees. f has a minimum at $x = 0$. But there is no maximum.

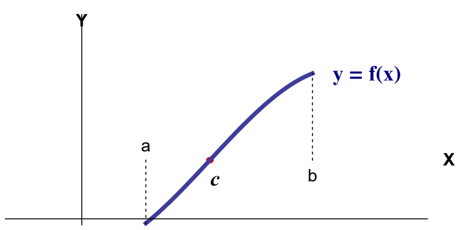
Theory Exercises (Optional- but read thoughtfully)

Prove each using a variation of the ***Extreme Value Theorem*** proof.

Existence of Zeros

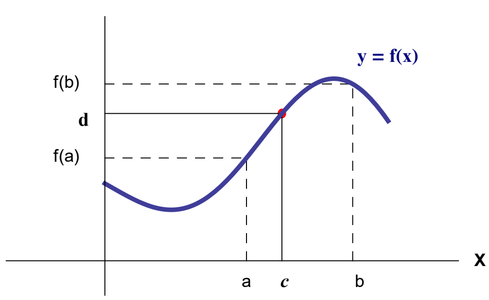
Finding the zeros of a function is important in almost any area of mathematics. This theo-rem says that a continuous function has zeros where you expect them, based on your knowledge of its graph. The theorem guarantees the existence of a zero, but does little to help you find it. You know a few methods of finding zeros such as the quadratic formula for quadratic equations. Calculus will provide a good method
(Newton's Method) of finding zeros as accurately as you wish for a wide variety of functions. What this theorem does is tell you when it is guaranteed worth while spending time looking for a zero.

Existence of a Zero Theorem Let f be continuous on the closed interval $a \leq x \leq b$. Suppose f(a) and f(b) have opposite signs. Then there exists a number c, $a < c < b$, such that $f(c) = 0$.

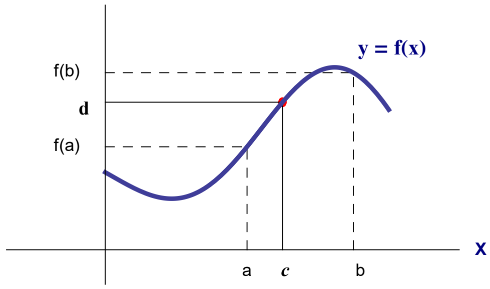


Intermediate Value Theorem

This theorem is a generalization of the Existence of a Zero Theorem. It says a continuous function on the closed interval $a \leq x \leq b$ takes on all values between f(a) and f(b).

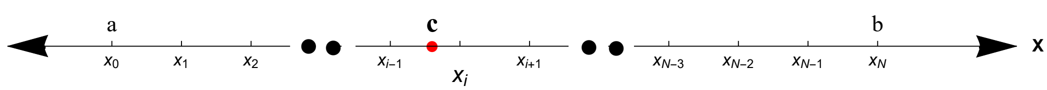


This theorem is a generalization of the Existence of a Zero Theorem. It says a continuous function on the interval $a \le x \le b$ takes on all values between f(a) and f(b). The proof is very similar to the proof of the previous theorem and is left as an exercise.



Intermediate Value Theorem Let f be continuous on the closed interval $a \le x \le b$. Suppose d is a number between f(a) and f(b). Then there exists a number c, $a < c < b$, such that $f(c) = d$.

Proof Subdivide the interval $a \le x \le b$ into an infinite number N of subdivisions of infinitesimal length $\frac{b-a}{N}$ by the sequence $x_0, x_1, x_2, \cdots, x_N$.



Suppose $f(a) > 0$. Then compute the sequence of values $f(a), f(x_1), f(x_2), \cdots$ until $f(x_i) < 0$ (Closed Sequence Principle). Then by the continuity of f, recalling the first preliminary note, there is a real number c such that

$x_i \approx\!> c$ and $f(x_i) \approx\!> f(c) = 0$.

So we have found the real zero c exactly by a hyperreal calculation.

The proof of the existence of c in the case where $f(a) < 0$ is similar and is left for you.

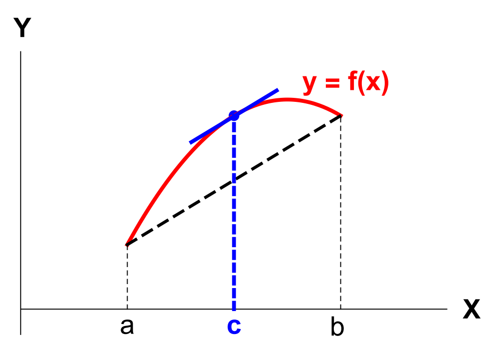
End of Proof

The Mean Value Theorem

The next section in many calculus textbooks is *The Mean Value Theorem (for Derivatives)*. It is used to prove theorems later in calculus. We will not need it for this textbook. Most students find the theory tedious and hard to understand its use in proofs; an intuitive understanding or other approach is better. So we will simply state it. If you need in a later calculus based course, the instructor will review it because it will have been totally forgotten by most students.

The Mean Value Theorem Let f be continuous for $a \leqslant x \leqslant b$.
Let f be differentiable for $a < x < b$. Then there is a c, $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



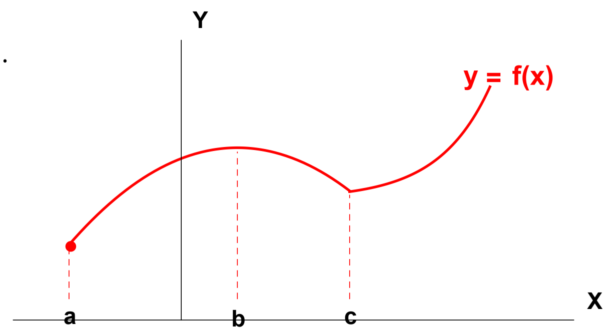
Note This theorem is also called the 'Mean Value Theorem for Derivatives.'
It states that under the hypotheses, 'There is at least one point c in the interval where the slope of the curve is the same as the slope of the line joining the endpoints.'

3.2 The Extreme Values of a Function

Given any quantity described by a function, we are often interested in the largest and/or smallest values that quantity attains. For instance, if a function describes the speed of an object, it seems reasonable to want to know the fastest/slowest the object traveled. If a function describes the value of a stock, we might want to know the highest/lowest values the stock attained over the past year. We call such values *extreme values.*

In Section 3.1, in the Extreme Value Theorem, we talked about the *extreme values* of a function, namely its *maximum* and *minimum*. But in the process of finding these extreme values, we will see that the situation is somewhat complicated and we need to further clarify the terms maximum and minimum. Let us look at an example.

Example



The minimum value of the function f is clearly at x = a. We will call the minimum at x = a the ***global minimum*** of the function.

But there is some kind of minimum at x = c even though it is not the least value on the domain of f. However, it is the minimum over values of x infinitesimally close to c. We will call this minimum a ***local minimum*** of f.
Note that x = a is also a local minimum of f.

f has a l*ocal maximum* at x = b. It does not have *global maximum.*

NOTE Some applied mathematicians say f has a global maximum at x = +∞. We will not.

Definition 3.2.1 Extreme Values

Let f be defined on an interval I containing c.

1. $f(c)$ is the **global minimum** of f on I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **global maximum** of f on I if $f(c) \geq f(x)$ for all x in I.

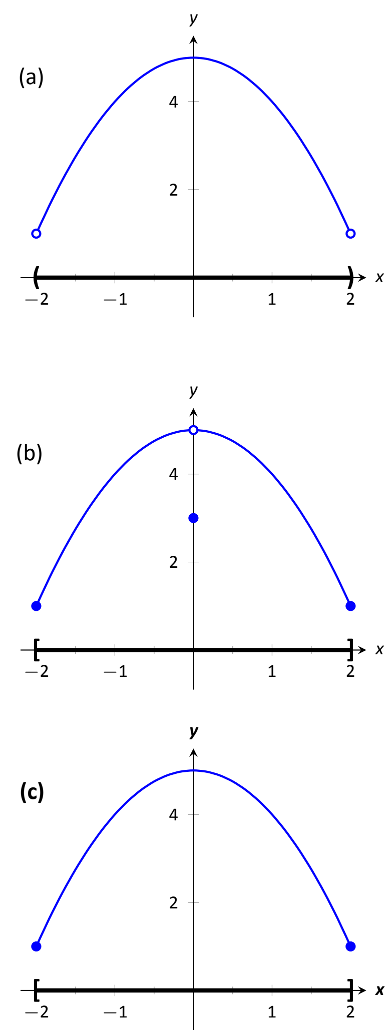


Figure 3.2.1: Graphs of functions with and without extreme values.

The function displayed in (a) has a maximum, but no minimum, as the interval over which the function is defined is open. In (b), the function has a minimum, but no maximum; there is a discontinuity in the "natural" place for the maximum to occur. Finally, the function shown in (c) has both a maximum and a minimum; note that the function is continuous and the interval on which it is defined is closed.

It is possible for discontinuous functions defined on an open interval to have both a maximum and minimum value, but we have just seen examples where they did not. On the other hand, continuous functions on a closed interval *always* have a maximum and minimum value.

In most applications we are interested in finding the Global Extreme Values, not the Local Extreme Values.

The importance of local extrema is they are relatively easy to find using calculus methods. Then, in a very important case, it is easy to find the global extreme values. In other cases graphical methods work rather well.

Local Extreme Values

Note: The extreme values of a function are "*y*" values, values the function attains, not the "x" values.

Locating Theorem for Local Extrema

They *may* occur *only*:

1. **where f'(x) = 0**
2. **where f'(x) does not exist.**
3. **at endpoints.**

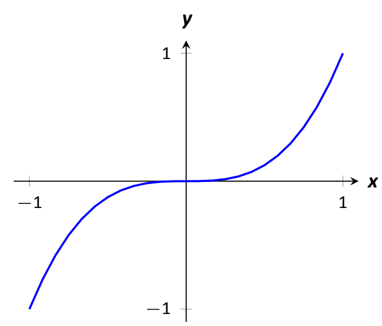
Proof

I. **f'(x) exists. If f'(x) ≠ 0, then f is increasing or decreasing ⟺ no local extremum. So f'(x) = 0.**

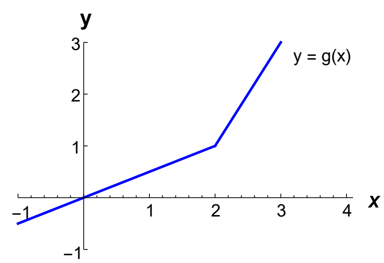
II. **f'(x) DNE ⟺ 2 or 3 (ignoring any endpoint agreement).**

Apex calls 1 and 2 ***critical values***.

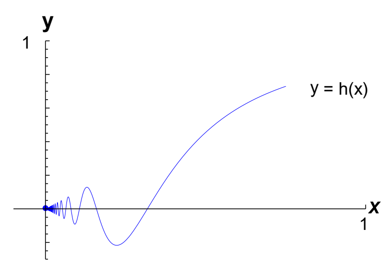
Note: f '(a) = 0 does not not necessarily mean there is a local extreme value at x = 0.



Note: g '(2) DNE does not not necessarily mean there is a local extreme value at x = 2.



Note: there usually is a local extreme value at an endpoint. However, there is no local extreme value at x =0 in this example.



Extreme Values **Finding Extrema of a Continuous Function on a Closed Interval**

1. Find all local extrema on the interval using the Locating Theorem.
2. The least value is the global minimum.
   The greatest value is the global maximum.

This is because the Extreme Value Theorem says the extreme values exist.

We practice the above ideas in the next examples.

Example 3.2.4 Finding extreme values

Find the extreme values of $f(x) = 2x^3 + 3x^2 - 12x$ on $[0, 3]$, graphed in Figure 3.1.6(a).

Solution

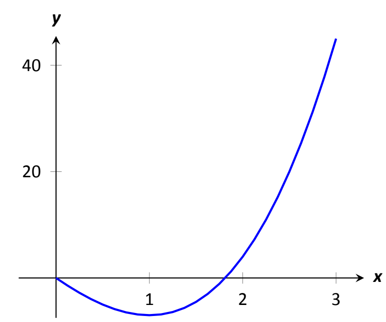
We follow the steps outlined above. We first evaluate f at the endpoints:

$$f(0) = 0 \quad \text{and} \quad f(3) = 45.$$

Next, we find the critical values of f on $[0, 3]$. $f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$; therefore the critical values of f are $x = -2$ and $x = 1$. Since $x = -2$ does not lie in the interval $[0, 3]$, we ignore it. Evaluating f at the only critical number in our interval gives: $f(1) = -7$.

The table in Figure 3.2.6(b) gives f evaluated at the "important" x values in $[0, 3]$. We can easily see the maximum and minimum values of f: the maximum value is 45 and the minimum value is -7.

Note that all this was done without the aid of a graph; this work followed an analytic algorithm and did not depend on any visualization. Figure 3.2.6 shows f and we can confirm our answer, but it is important to understand that these answers can be found without graphical assistance.



(a)

x	$f(x)$
0	0
1	-7
3	45

(b)

Figure 3.2.6: Finding the extreme values of $f(x) = 2x^3 + 3x^2 - 12x$ in Example 3.1.4.

We practice again.

Example 3.2.5 **Finding extreme values** Find the maximum and minimum values of f on $[-4, 2]$, where

$$f'(x) = \begin{cases} 2(x-1) & x < 0 \\ 1 & x > 0 \end{cases}$$

graphed in Figure 3.2.7(a).

Solution Here f is piecewise–defined, but we can still apply Key Idea 3.1.1 as it is continuous on $[-4, 2]$ (one should check to verify that $\lim_{x\to 0} f(x) = \mathrm{f}(0)$).

Evaluating f at the endpoints gives:

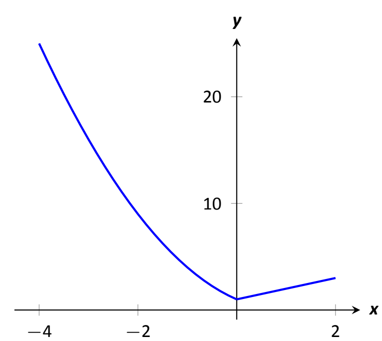
$$f(-4) = 25 \quad \text{and} \quad f(2) = 3.$$

We now find the critical numbers of f. We have to define f' in a piecewise manner; it is

$$f'(x) = \begin{cases} 2(x-1) & x < 0 \\ 1 & x > 0 \end{cases}.$$

Note that while f is defined for all of $[-4, 2]$, f' is not, as the derivative of f does not exist when $x = 0$. (From the left, the derivative approaches -2; from the right the derivative is 1.) Thus one critical number of f is $x = 0$.

We now set $f'(x) = 0$. When $x > 0$, $f'(x)$ is never 0. When $x < 0$, $f'(x)$ is also never 0, so we find no critical values from setting $f'(x) = 0$.
So we have three important x values to consider: $x = -4$, 2 and 0. Evaluating f at each gives, respectively, 25, 3 and 1, shown in Figure 3.2.7(b). Thus the

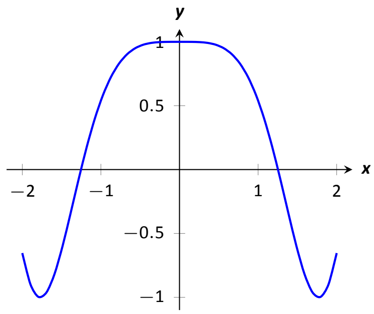


(a)

x	$f(x)$
-4	25
0	1
2	3

(b)

Figure 3.2.7: Finding the extreme values of a piecewise–defined function in Example 3.2.5.



(a)

x	$f(x)$
-2	-0.65
$-\sqrt{\pi}$	-1
0	1
$\sqrt{\pi}$	-1
2	-0.65

(b)

Figure 3.2.8: Finding the extrema of $f(x) = \cos(x^2)$ in Example 3.2.6.

absolute minimum of f is 1, the absolute maximum of f is 25, confirmed by the graph of f.

Example 3.2.6 Finding extreme values

Find the extrema of $f(x) = \cos(x^2)$ on $[-2, 2]$, graphed in Figure 3.2.8(a).

Solution Evaluating f at the endpoints of the interval gives: $f(-2) = f(2) = \cos(4) \approx -0.6536$. We now find the critical values of f.

Applying the Chain Rule, we find $f'(x) = -2x\sin(x^2)$. Set $f'(x) = 0$ and solve for x to find the critical values of f.

We have $f'(x) = 0$ when $x = 0$ and when $\sin(x^2) = 0$. In general, $\sin t = 0$ when $t = \ldots - 2\pi, -\pi, 0, \pi, \ldots$ Thus $\sin(x^2) = 0$ when $x^2 = 0, \pi, 2\pi, \ldots$ (x^2 is always positive so we ignore $-\pi$, etc.) So $\sin(x^2) = 0$ when $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}$, etc. The only values to fall in the given interval of $[-2, 2]$ are 0 and $\pm\sqrt{\pi}$, where $\sqrt{\pi} \doteq 1.77$.

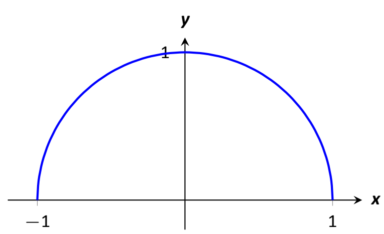
We again construct a table of important values in Figure 3.2.8(b). In this example we have 5 values to consider: $x = 0, \pm 2, \pm\sqrt{\pi}$.

From the table it is clear that the maximum value of f on $[-2, 2]$ is 1; the minimum value is -1. The graph of f confirms our results.

We consider one more example.

Example 3.2.7 Finding extreme values

Find the extreme values of $f(x) = \sqrt{1-x^2}$, graphed in Figure 3.2.9(a).



(a)

x	$f(x)$
-1	0
0	1
1	0

(b)

Figure 3.2.9: Finding the extrema of the half–circle in Example 3.2.7.

Note: We implicitly found the derivative of $x^2 + y^2 = 1$, the unit circle, in Example 2.6.5 as $\frac{dy}{dx} = -x/y$. In Example 3.1.7, half of the unit circle is given as $y = f(x) = \sqrt{1-x^2}$. We found $f'(x) = \frac{-x}{\sqrt{1-x^2}}$. Recognize that the denominator of this fraction is y; that is, we again found $f'(x) = \frac{dy}{dx} = -x/y$.

Solution A closed interval is not given, so we find the extreme values of f on its domain. f is defined whenever $1 - x^2 \geq 0$; thus the domain of f is $[-1, 1]$. Evaluating f at either endpoint returns 0.

Using the Chain Rule, we find $f'(x) = \dfrac{-x}{\sqrt{1-x^2}}$. The critical points of f are found when $f'(x) = 0$ or when f' is undefined. It is straightforward to find that $f'(x) = 0$ when $x = 0$, and f' is undefined when $x = \pm 1$, the endpoints of the interval. The table of important values is given in Figure 3.2.9(b). The maximum value is 1, and the minimum value is 0. (See also the marginal note.)

We have seen that continuous functions on closed intervals always have a maximum and minimum value, and we have also developed a technique to find these values. In the next section, we further our study of the information we can glean from "nice" functions with the Mean Value Theorem. On a closed interval, we can find the *average rate of change* of a function (as we did at the beginning of Chapter 2). We will see that differentiable functions always have a point at which their *instantaneous* rate of change is same as the *average* rate of change. This is surprisingly useful, as we'll see.

Global Extreme Values. Other Cases

1. Find all local extrema using the Locating Theorem.
2. Graph by hand or a CAS.
3. Choose the global extreme values from 1.

Exercises 3.2

Terms and Concepts

1. Describe what an "extreme value" of a function is in your own words.

2. Sketch the graph of a function f on $(-1,1)$ that has both a maximum and minimum value.

3. Describe the difference between absolute and relative maxima in your own words.

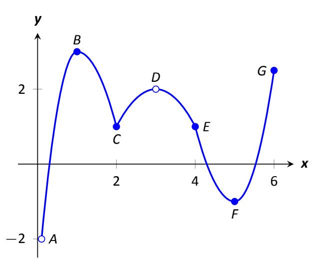
4. Sketch the graph of a function f where f has a relative maximum at $x=1$ and $f'(1)$ is undefined.

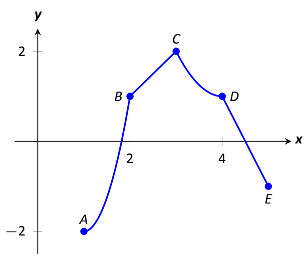
5. T/F: If c is a critical value of a function f, then f has either a relative maximum or relative minimum at $x=c$.

6. Fill in the blanks: The critical points of a function f are found where $f'(x)$ is equal to ________ or where $f'(x)$ is ________.

Problems

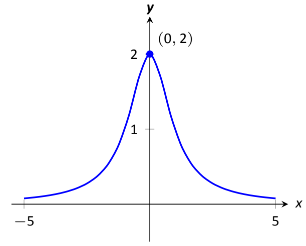
In Exercises 7 – 8, identify each of the marked points as being an global maximum or minimum, a local maximum or minimum, or none of the above.

7. 

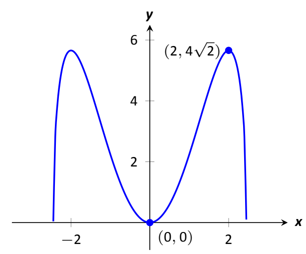
8. 

In Exercises 9 – 16, evaluate $f'(x)$ at the points indicated in the graph.

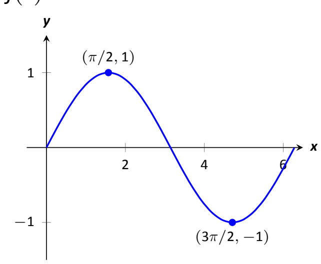
9. $f(x) = \dfrac{2}{x^2+1}$



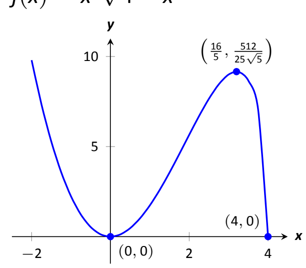
10. $f(x) = x^2\sqrt{6-x^2}$



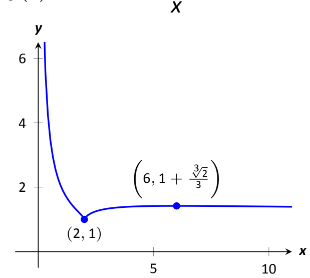
11. $f(x) = \sin x$



12. $f(x) = x^2\sqrt{4-x}$



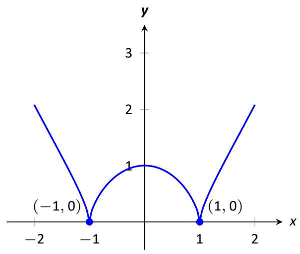
13. $f(x) = 1 + \dfrac{(x-2)^{2/3}}{x}$



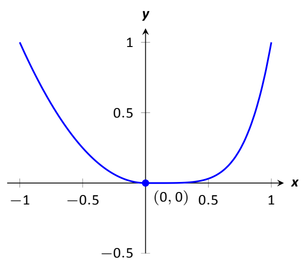
Solutions 3.2

1. Answers will vary.
2. Answers will vary.
3. Answers will vary.
4. Answers will vary.
5. F
6. Where $f'(x)$ is equal to <u>0</u> or where $f'(x)$ is <u>undefined</u>.
7. A: none; the function isn't defined here. B: abs. max & rel. max C: rel. min D: none; the function isn't defined here. E: none F: rel. min G: rel. max
8. A: abs. min & rel. min B: none C: abs. max & rel. max D: none E: rel. min
9. $f'(0) = 0$
10. $f'(0) = 0\, f'(2) = 0$
11. $f'(\pi/2) = 0\, f'(3\pi/2) = 0$
12. $f'(0) = 0\, f'(3.2) = 0\, f'(4)$ is undefined
13. $f'(2)$ is not defined $f'(6) = 0$
14. Both $f'(-1)$ and $f'(1)$ are undefined.
15. $f'(0) = 0$
16. $f'(0)$ is not defined
17. min: $(-0.5, 3.75)$ max: $(2, 10)$

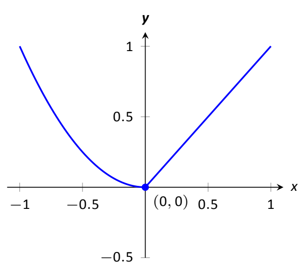
14. $f(x) = \sqrt[3]{x^4 - 2x + 1}$



15. $f(x) = \begin{cases} x^2 & x \leq 0 \\ x^5 & x > 0 \end{cases}$



16. $f(x) = \begin{cases} x^2 & x \leq 0 \\ x & x > 0 \end{cases}$



In Exercises 17 – 26, find the extreme values of the function on the given interval.

17. $f(x) = x^2 + x + 4$ on $[-1, 2]$.

18. $f(x) = x^3 - \frac{9}{2}x^2 - 30x + 3$ on $[0, 6]$.

19. $f(x) = 3\sin x$ on $[\pi/4, 2\pi/3]$.

20. $f(x) = x^2\sqrt{4 - x^2}$ on $[-2, 2]$.

21. $f(x) = x + \frac{3}{x}$ on $[1, 5]$.

22. $f(x) = \frac{x^2}{x^2 + 5}$ on $[-3, 5]$.

23. $f(x) = e^x \cos x$ on $[0, \pi]$.

24. $f(x) = e^x \sin x$ on $[0, \pi]$.

25. $f(x) = \frac{\ln x}{x}$ on $[1, 4]$.

26. $f(x) = x^{2/3} - x$ on $[0, 2]$.

Review

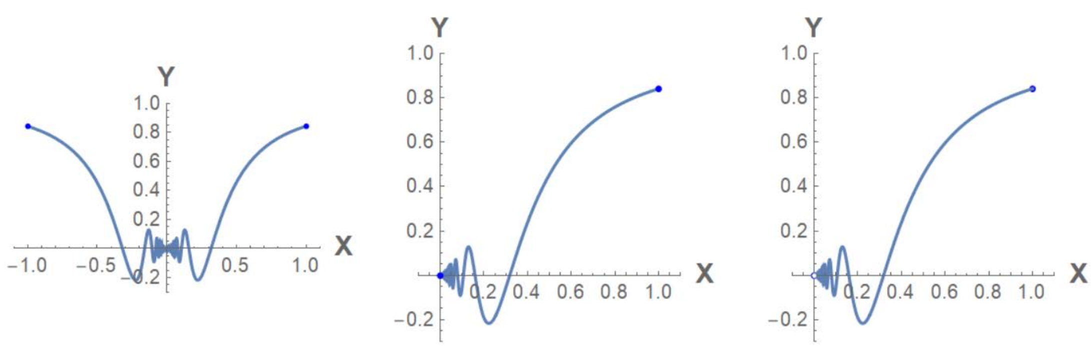
27. Find $\frac{dy}{dx}$, where $x^2y - y^2x = 1$.

28. Find the equation of the line tangent to the graph of $x^2 + y^2 + xy = 7$ at the point $(1, 2)$.

29. Let $f(x) = x^3 + x$.

Evaluate $\lim_{s \to 0} \frac{f(x + s) - f(x)}{s}$.

30. Identify approximately the global extreme values of each.



3.3 Increasing and Decreasing Functions

Our study of "nice" functions f in this chapter has so far focused on individual points: points where f is maximal/minimal, points where $f'(x) = 0$ or f' does not exist, and points c where $f'(c)$ is the average rate of change of f on some interval.

In this section we begin to study how functions behave *between* special points; we begin studying in more detail the shape of their graphs.

We start with an intuitive concept. Given the graph in Figure 3.3.1, where would you say the function is *increasing*? *Decreasing*? Even though we have not defined these terms mathematically, one likely answered that f is increasing when $x > 1$ and decreasing when $x < 1$. We formally define these terms here.

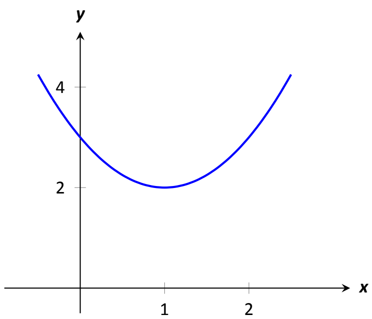


Figure 3.3.1: A graph of a function f used to illustrate the concepts of *increasing* and *decreasing*.

Definition 3.3.1 Increasing and Decreasing Functions

Let f be a function defined on an interval I.

1. f is **increasing** on I if for every $a < b$ in I, $f(a) < f(b)$.
2. f is **decreasing** on I if for every $a < b$ in I, $f(a) > f(b)$.

How can you tell an uphill hill from an downhill hill? Answer: it depends which way you are walking. In mathematics we make the determination by walking to the right.

Informally, a function is increasing if as x gets larger (i.e., looking left to right) $f(x)$ gets larger.

Our interest lies in finding intervals in the domain of f on which f is either increasing or decreasing. Such information should seem useful. For instance, if f describes the speed of an object, we might want to know when the speed was increasing or decreasing (i.e., when the object was accelerating vs. decelerating). If f describes the population of a city, we should be interested in when the population is growing or declining.

To find such intervals, we again consider secant lines. Let f be an increasing, differentiable function on an open interval I, such as the one shown in Figure 3.3.2, and let $a < b$ be given in I. The secant line on the graph of f from $x = a$ to $x = b$ is drawn; it has a slope of $(f(b) - f(a))/(b - a)$. But note:

$$\frac{f(b)-f(a)}{b-a} \Rightarrow \frac{\text{numerator } > 0}{\text{denominator } > 0} \Rightarrow \begin{array}{c}\text{slope of the}\\ \text{secant line} > 0\end{array} \Rightarrow \begin{array}{c}\text{Average rate of}\\ \text{change of } f \text{ on}\\ [a,b] \text{ is} > 0.\end{array}$$

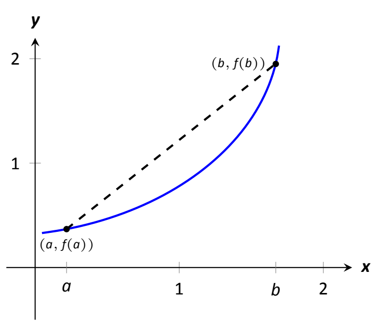


Figure 3.3.2: Examining the secant line of an increasing function.

By considering all such secant lines in I, we strongly imply that $f'(x) > 0$ on I. A similar statement can be made for decreasing functions.

Our above logic can be summarized as "If f is increasing, then f' is probably positive." Theorem 3.3.1 below turns this around by stating "If f' is positive, then f is increasing." This leads us to a method for finding when functions are increasing and decreasing.

Theorem 3.3.1 Test For Increasing/Decreasing Functions

Let f be a continuous function on $[a, b]$ and differentiable on (a, b).

1. If $f'(c) > 0$ for all c in (a, b), then f is increasing on $[a, b]$.
2. If $f'(c) < 0$ for all c in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(c) = 0$ for all c in (a, b), then f is constant on $[a, b]$.

Note: Parts 1 & 2 of Theorem 3.3.1 also hold if $f'(c) = 0$ for a finite number of values of c in I.

Let f be differentiable on an interval I and let a and b be in I where $f'(a) > 0$ and $f'(b) < 0$. If f' is continuous on $[a, b]$, it follows from the Intermediate Value Theorem that there must be some value c between a and b where $f'(c) = 0$. (It turns out that this is still true even if f' is not continuous on $[a, b]$.) This leads us to the following method for finding intervals on which a function is increasing or decreasing.

Key Idea 3.3.1 Finding Intervals on Which f is Increasing or Decreasing

Let f be a differentiable function on an interval I. To find intervals on which f is increasing and decreasing:

1. Find the critical values of f. That is, find all c in I where $f'(c) = 0$ or f' is not defined.
2. Use the critical values to divide I into subintervals.
3. Pick any point p in each subinterval, and find the sign of $f'(p)$.
   (a) If $f'(p) > 0$, then f is increasing on that subinterval.
   (b) If $f'(p) < 0$, then f is decreasing on that subinterval.

We demonstrate using this process in the following example.

Example 3.3.1 Finding intervals of increasing/decreasing

Let $f(x) = x^3 + x^2 - x + 1$. Find intervals on which f is increasing or decreasing.

Solution Using Key Idea 3.3.1, we first find the critical values of f. We have $f'(x) = 3x^2 + 2x - 1 = (3x-1)(x+1)$, so $f'(x) = 0$ when $x = -1$ and when $x = 1/3$. f' is never undefined.

Since an interval was not specified for us to consider, we consider the entire domain of f which is $(-\infty, \infty)$. We thus break the whole real line into three subintervals based on the two critical values we just found: $(-\infty, -1)$, $(-1, 1/3)$ and $(1/3, \infty)$. This is shown in Figure 3.3.3.

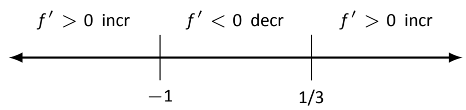


Figure 3.3.3: Number line for f in Example 3.3.1.

We now pick a value p in each subinterval and find the sign of $f'(p)$. All we care about is the sign, so we do not actually have to fully compute $f'(p)$; pick "nice" values that make this simple.

Subinterval 1, $(-\infty, -1)$: We (arbitrarily) pick $p = -2$. We can compute $f'(-2)$ directly: $f'(-2) = 3(-2)^2 + 2(-2) - 1 = 7 > 0$. We conclude that f is increasing on $(-\infty, -1)$.

Note we can arrive at the same conclusion without computation. For instance, we could choose $p = -100$. The first term in $f'(-100)$, i.e., $3(-100)^2$ is clearly positive and very large. The other terms are small in comparison, so we know $f'(-100) > 0$. All we need is the sign.

Subinterval 2, $(-1, 1/3)$: We pick $p = 0$ since that value seems easy to deal with. $f'(0) = -1 < 0$. We conclude f is decreasing on $(-1, 1/3)$.

Subinterval 3, $(1/3, \infty)$: Pick an arbitrarily large value for $p > 1/3$ and note that $f'(p) = 3p^2 + 2p - 1 > 0$. We conclude that f is increasing on $(1/3, \infty)$.

We can verify our calculations by considering Figure 3.3.4, where f is graphed. The graph also presents f'; note how $f' > 0$ when f is increasing and $f' < 0$ when f is decreasing.

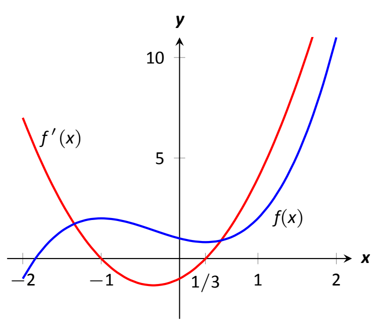


Figure 3.3.4: A graph of $f(x)$ in Example 3.3.1, showing where f is increasing and decreasing.

One is justified in wondering why so much work is done when the graph seems to make the intervals very clear. We give three reasons why the above work is worthwhile.

First, the points at which f switches from increasing to decreasing are not precisely known given a graph. The graph shows us something significant happens near $x = -1$ and $x = 0.3$, but we cannot determine exactly where from the graph.

One could argue that just finding critical values is important; once we know the significant points are $x = -1$ and $x = 1/3$, the graph shows the increasing/decreasing traits just fine. That is true. However, the technique prescribed here helps reinforce the relationship between increasing/decreasing and the sign of f'. Once mastery of this concept (and several others) is obtained, one finds that either (a) just the critical points are computed and the graph shows all else that is desired, or (b) a graph is never produced, because determining increasing/decreasing using f' is straightforward and the graph is unnecessary. So our second reason why the above work is worthwhile is this: once mastery of a subject is gained, one has *options* for finding needed information. We are working to develop mastery.

Finally, our third reason: many problems we face "in the real world" are very complex. Solutions are tractable only through the use of computers to do many calculations for us. Computers do not solve problems "on their own," however; they need to be taught (i.e., *programmed*) to do the right things. It would be beneficial to give a function to a computer and have it return maximum and minimum values, intervals on which the function is increasing and decreasing, the locations of relative maxima, etc. The work that we are doing here is easily programmable. It is hard to teach a computer to "look at the graph and see if it is going up or down." It is easy to teach a computer to "determine if a number is greater than or less than 0."

In Section 3.1 we learned the definition of relative maxima and minima and found that they occur at critical points. We are now learning that functions can switch from increasing to decreasing (and vice–versa) at critical points. This new understanding of increasing and decreasing creates a great method of determining whether a critical point corresponds to a maximum, minimum, or neither. Imagine a function increasing until a critical point at $x = c$, after which it decreases. A quick sketch helps confirm that $f(c)$ must be a relative maximum. A similar statement can be made for relative minimums. We formalize this concept in a theorem.

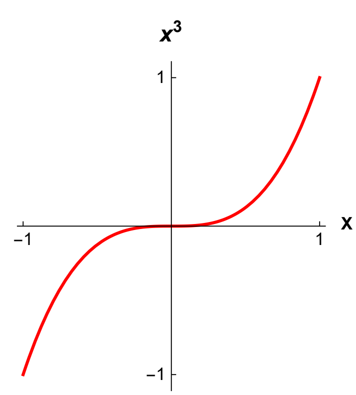
Theorem 3.3.2 First Derivative Test

Let f be differentiable on an interval I and let c be a critical number in I.

1. If the sign of f' switches from positive to negative at c, then $f(c)$ is a relative maximum of f.
2. If the sign of f' switches from negative to positive at c, then $f(c)$ is a relative minimum of f.
3. If f' is positive (or, negative) before and after c, then $f(c)$ is not a relative extrema of f.

Everyone should understand this theorem even though it is not very efficient in determining the nature of a critical point.

Case 3, such as the function $y = x^3$ at $x = 0$, is sometimes referred to as ***stationary point*** since the y-value does not change much near the point.



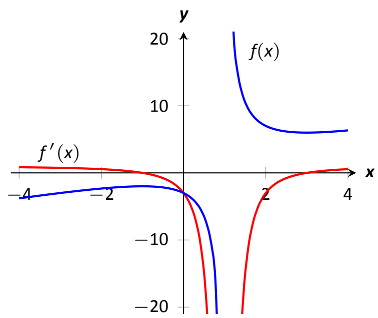


Figure 3.3.5: A graph of $f(x)$ in Example 3.3.2, showing where f is increasing and decreasing.

Example 3.3.2 Using the First Derivative Test

Find the intervals on which f is increasing and decreasing, and use the First Derivative Test to determine the relative extrema of f, where

$$f(x) = \frac{x^2+3}{x-1}.$$

Solution We start by noting the domain of f: $(-\infty, 1) \cup (1, \infty)$. Key Idea 3.3.1 describes how to find intervals where f is increasing and decreasing *when the domain of f is an interval.* Since the domain of f in this example is the union of two intervals, we apply the techniques of Key Idea 3.3.1 to both intervals of the domain of f.

Since f is not defined at $x = 1$, the increasing/decreasing nature of f could switch at this value. We do not formally consider $x = 1$ to be a critical value of f, but we will include it in our list of critical values that we find next.

Using the Quotient Rule, we find

$$f'(x) = \frac{x^2-2x-3}{(x-1)^2}.$$

We need to find the critical values of f; we want to know when $f'(x) = 0$ and when f' is not defined. That latter is straightforward: when the denominator of $f'(x)$ is 0, f' is undefined. That occurs when $x = 1$, which we've already recognized as an important value.

$f'(x) = 0$ when the numerator of $f'(x)$ is 0. That occurs when $x^2 - 2x - 3 = (x-3)(x+1) = 0$; i.e., when $x = -1, 3$.

We have found that f has two critical numbers, $x = -1, 3$, and at $x = 1$ something important might also happen. These three numbers divide the real number line into 4 subintervals:

$$(-\infty, -1), \quad (-1, 1), \quad (1, 3) \quad \text{and} \quad (3, \infty).$$

Pick a number p from each subinterval and test the sign of f' at p to determine whether f is increasing or decreasing on that interval. Again, we do well to avoid complicated computations; notice that the denominator of f' is *always* positive so we can ignore it during our work.

Interval 1, $(-\infty, -1)$: Choosing a very small number (i.e., a negative number with a large magnitude) p returns $p^2 - 2p - 3$ in the numerator of f'; that will be positive. Hence f is increasing on $(-\infty, -1)$.

Interval 2, $(-1, 1)$: Choosing 0 seems simple: $f'(0) = -3 < 0$. We conclude f is decreasing on $(-1, 1)$.

Interval 3, $(1, 3)$: Choosing 2 seems simple: $f'(2) = -3 < 0$. Again, f is decreasing.

If you think this page and the following two are interesting, get a life.

If you need it to distinguish a mountain top from a valley bottom, don't take up hiking.

Interval 4, $(3, \infty)$: Choosing an very large number p from this subinterval will give a positive numerator and (of course) a positive denominator. So f is increasing on $(3, \infty)$.

In summary, f is increasing on the intervals $(-\infty, -1)$ and $(3, \infty)$ and is decreasing on the intervals $(-1, 1)$ and $(1, 3)$. Since at $x = -1$, the sign of f' switched from positive to negative, Theorem 3.3.2 states that $f(-1)$ is a relative maximum of f. At $x = 3$, the sign of f' switched from negative to positive, meaning $f(3)$ is a relative minimum. At $x = 1$, f is not defined, so there is no relative extrema at $x = 1$.

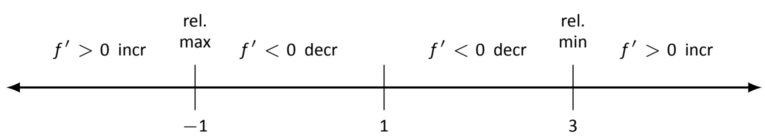


Figure 3.3.6: Number line for f in Example 3.3.2.

This is summarized in the number line shown in Figure 3.3.6. Also, Figure 3.3.5 shows a graph of f, confirming our calculations. This figure also shows f', again demonstrating that f is increasing when $f' > 0$ and decreasing when $f' < 0$.

One is often tempted to think that functions always alternate "increasing, decreasing, increasing, decreasing,. . ." around critical values. Our previous example demonstrated that this is not always the case. While $x = 1$ was not technically a critical value, it was an important value we needed to consider. We found that f was decreasing on "both sides of $x = 1$."

We examine one more example.

Example 3.3.3 Using the First Derivative Test

Find the intervals on which $f(x) = x^{8/3} - 4x^{2/3}$ is increasing and decreasing and identify the relative extrema.

SOLUTION We again start with taking a derivative. Since we know we want to solve $f'(x) = 0$, we will do some algebra after taking the derivative.

$$\begin{aligned} f(x) &= x^{\frac{8}{3}} - 4x^{\frac{2}{3}} \\ f'(x) &= \frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3}x^{-\frac{1}{3}} \\ &= \frac{8}{3}x^{-\frac{1}{3}}\left(x^{\frac{6}{3}} - 1\right) \end{aligned}$$

$$= \frac{8}{3}x^{-\frac{1}{3}}(x^2-1)$$
$$= \frac{8}{3}x^{-\frac{1}{3}}(x-1)(x+1).$$

This derivation of f' shows that $f'(x) = 0$ when $x = \pm 1$ and f' is not defined when $x = 0$. Thus we have 3 critical values, breaking the number line into 4 subintervals as shown in Figure 3.3.7.

Interval 1, $(\infty, -1)$: We choose $p = -2$; we can easily verify that $f'(-2) < 0$. So f is decreasing on $(-\infty, -1)$.
Interval 2, $(-1, 0)$: Choose $p = -1/2$. Once more we practice finding the sign of $f'(p)$ without computing an actual value. We have $f'(p) = (8/3)p^{-1/3}(p-1)(p+1)$; find the sign of each of the three terms.

$$f'(p) = \frac{8}{3} \cdot \underbrace{p^{-\frac{1}{3}}}_{<0} \cdot \underbrace{(p-1)}_{<0}\underbrace{(p+1)}_{>0}.$$

We have a "negative $\times$ negative $\times$ positive" giving a positive number; f is increasing on $(-1, 0)$.
Interval 3, $(0, 1)$: We do a similar sign analysis as before, using p in $(0, 1)$.

$$f'(p) = \frac{8}{3} \cdot \underbrace{p^{-\frac{1}{3}}}_{>0} \cdot \underbrace{(p-1)}_{<0}\underbrace{(p+1)}_{>0}.$$

We have 2 positive factors and one negative factor; $f'(p) < 0$ and so f is decreasing on $(0, 1)$.
Interval 4, $(1, \infty)$: Similar work to that done for the other three intervals shows that $f'(x) > 0$ on $(1, \infty)$, so f is increasing on this interval.

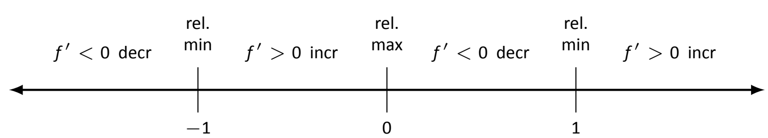


Figure 3.3.7: Number line for f in Example 3.3.3.

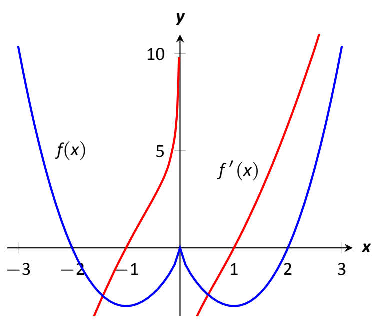


Figure 3.3.8: A graph of $f(x)$ in Example 3.3.3, showing where f is increasing and decreasing.

We conclude by stating that f is increasing on the intervals $(-1, 0)$ and $(1, \infty)$ and decreasing on the intervals $(-\infty, -1)$ and $(0, 1)$. The sign of f' changes from negative to positive around $x = -1$ and $x = 1$, meaning by Theorem 3.3.2 that $f(-1)$ and $f(1)$ are relative minima of f. As the sign of f' changes from positive to negative at $x = 0$, we have a relative maximum at $f(0)$. Figure 3.3.8 shows a graph of f, confirming our result. We also graph f', highlighting once more that f is increasing when $f' > 0$ and is decreasing when $f' < 0$.

We have seen how the first derivative of a function helps determine when the function is going "up" or "down." In the next section, we will see how the second derivative helps determine how the graph of a function curves.

If you work all the exercises, you might not finish before graduation.

Exercises 3.3

Terms and Concepts

1. In your own words describe what it means for a function to be increasing.

2. What does a decreasing function "look like"?

3. Sketch a graph of a function on $[0, 2]$ that is increasing, where it is increasing "quickly" near $x = 0$ and increasing "slowly" near $x = 2$.

4. Give an example of a function describing a situation where it is "bad" to be increasing and "good" to be decreasing.

5. T/F: Functions always switch from increasing to decreasing, or decreasing to increasing, at critical points.

6. A function f has derivative $f'(x) = (\sin x + 2)e^{x^2+1}$, where $f'(x) > 1$ for all x. Is f increasing, decreasing, or can we not tell from the given information?

Problems

In Exercises 7 – 14, a function $f(x)$ is given.

(a) Compute $f'(x)$.

(b) Graph f and f' on the same axes (using technology is permitted) and verify Theorem 3.3.1.

7. $f(x) = 2x + 3$

8. $f(x) = x^2 - 3x + 5$

9. $f(x) = \cos x$

10. $f(x) = \tan x$

11. $f(x) = x^3 - 5x^2 + 7x - 1$

12. $f(x) = 2x^3 - x^2 + x - 1$

13. $f(x) = x^4 - 5x^2 + 4$

14. $f(x) = \dfrac{1}{x^2 + 1}$

In Exercises 15 – 24, a function $f(x)$ is given.

Do one or two of these.

(a) Give the domain of f.

(b) Find the critical numbers of f.

(c) Create a number line to determine the intervals on which f is increasing and decreasing.

(d) Use the First Derivative Test to determine whether each critical point is a relative maximum, minimum, or neither.

15. $f(x) = x^2 + 2x - 3$

16. $f(x) = x^3 + 3x^2 + 3$

17. $f(x) = 2x^3 + x^2 - x + 3$

18. $f(x) = x^3 - 3x^2 + 3x - 1$

19. $f(x) = \dfrac{1}{x^2 - 2x + 2}$

20. $f(x) = \dfrac{x^2 - 4}{x^2 - 1}$

21. $f(x) = \dfrac{x}{x^2 - 2x - 8}$

22. $f(x) = \dfrac{(x - 2)^{2/3}}{x}$

23. $f(x) = \sin x \cos x$ on $(-\pi, \pi)$.

24. $f(x) = x^5 - 5x$

25. Give a graphical of a function for which
    1. f'(c) = 0, no extreme value
    2. f'(c) DNE, no extreme value
    3. c an endpoint, no extreme value.

Solutions 3.3

1. Answers will vary.
2. Answers will vary.
3. Answers will vary; graphs should be steeper near $x = 0$ than near $x = 2$.
4. Answers will vary.
5. False; for instance, $y = x^3$ is always increasing though it has a critical point at $x = 0$.
6. Increasing
7. Graph and verify.
8. Graph and verify.
9. Graph and verify.
10. Graph and verify.
11. Graph and verify.
12. Graph and verify.
13. Graph and verify.
14. Graph and verify.
15. domain: $(-\infty, \infty)$
    c.p. at $c = -1$;
    decreasing on $(-\infty, -1)$;
    increasing on $(-1, \infty)$;
    rel. min at $x = -1$.
16. domain=$(-\infty, \infty)$
    c.p. at $c = -2, 0$;
    increasing on $(-\infty, -2)$ and $(0, \infty)$;
    decreasing on $(-2, 0)$;
    rel. min at $x = 0$;
    rel. max at $x = -2$.
17. domain=$(-\infty, \infty)$
    c.p. at $c = \frac{1}{6}(-1 \pm \sqrt{7})$;
    decreasing on $(\frac{1}{6}(-1-\sqrt{7}), \frac{1}{6}(-1+\sqrt{7})))$;
    increasing on $(-\infty, \frac{1}{6}(-1-\sqrt{7}))$ and $(\frac{1}{6}(-1+\sqrt{7}), \infty)$;
    rel. min at $x = \frac{1}{6}(-1+\sqrt{7})$;
    rel. max at $x = \frac{1}{6}(-1-\sqrt{7})$.
18. domain=$(-\infty, \infty)$
    c.p. at $c = 1$;
    increasing on $(-\infty, \infty)$;
19. domain=$(-\infty, \infty)$
    c.p. at $c = 1$;
    decreasing on $(1, \infty)$
    increasing on $(-\infty, 1)$;
    rel. max at $x = 1$.
20. domain=$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
    c.p. at $c = 0$;
    decreasing on $(-\infty, -1)$ and $(-1, 0)$;
    increasing on $(0, 1)$ and $(1, \infty)$;
    rel. min at $x = 0$;
21. domain=$(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$
    no c.p.;
    decreasing on entire domain, $(-\infty, -2)$, $(-2, 4)$ and $(4, \infty)$
22. domain=$(-\infty, 0) \cup (0, \infty)$;
    c.p. at $c = 2, 6$;
    decreasing on $(-\infty, 0)$, $(0, 2)$ and $(6, \infty)$;
    increasing on $(2, 6)$;
    rel. min at $x = 2$; rel. max at $x = 6$.
23. domain=$(-\infty, \infty)$
    c.p. at $c = -3\pi/4, -\pi/4, \pi/4, 3\pi/4$;
    decreasing on $(-3\pi/4, -\pi/4)$ and $(\pi/4, 3\pi/4)$;
    increasing on $(-\pi, -3\pi/4)$, $(-\pi/4, \pi/4)$ and $(3\pi/4, \pi)$;
    rel. min at $x = -\pi/4, 3\pi/4$;
    rel. max at $x = -3\pi/4, \pi/4$.
24. domain = $(-\infty, \infty)$;
    c.p. at $c = -1, 1$;
    decreasing on $(-1, 1)$;
    increasing on $(-\infty, -1)$ and $(1, \infty)$;
    rel. min at $x = 1$;
    rel. max at $x = -1$

3.4 Concavity and the Second Derivative

Our study of "nice" functions continues. The previous section showed how the first derivative of a function, f', can relay important information about f. We now apply the same technique to f' itself, and learn what this tells us about f.

The key to studying f' is to consider its derivative, namely f'', which is the second derivative of f. When $f'' > 0$, f' is increasing. When $f'' < 0$, f' is decreasing. f' has relative maxima and minima where $f'' = 0$ or is undefined.

This section explores how knowing information about f'' gives information about f.

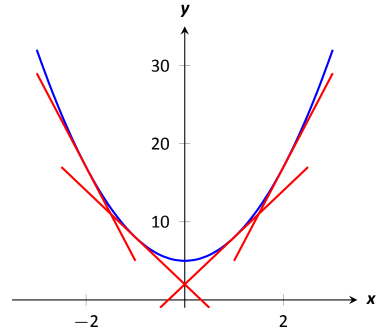


Figure 3.4.1: A function f with a concave up graph. Notice how the slopes of the tangent lines, when looking from left to right, are increasing.

Concavity

We begin with a definition, then explore its meaning.

Definition 3.4.1 Concave Up and Concave Down

Let f be differentiable on an interval I. The graph of f is **concave up** on I if f' is increasing. The graph of f is **concave down** on I if f' is decreasing. If f' is constant then the graph of f is said to have **no concavity**.

Note: We often state that "f is concave up" instead of "the graph of f is concave up" for simplicity.

The graph of a function f is *concave up* when f' is *increasing.* That means as one looks at a concave up graph from left to right, the slopes of the tangent lines will be increasing. Consider Figure 3.4.1, where a concave up graph is shown along with some tangent lines. Notice how the tangent line on the left is steep, downward, corresponding to a small value of f'. On the right, the tangent line is steep, upward, corresponding to a large value of f'.

Note: A mnemonic for remembering what concave up/down means is: "Concave up is like a cup; concave down is like a frown." It is admittedly terrible, but it works.

If a function is decreasing and concave up, then its rate of decrease is slowing; it is "leveling off." If the function is increasing and concave up, then the *rate* of increase is increasing. The function is increasing at a faster and faster rate.

Now consider a function which is concave down. We essentially repeat the above paragraphs with slight variation.

The graph of a function f is *concave down* when f' is *decreasing.* That means as one looks at a concave down graph from left to right, the slopes of the tangent lines will be decreasing. Consider Figure 3.4.2, where a concave down graph is shown along with some tangent lines. Notice how the tangent line on the left is steep, upward, corresponding to a large value of f'. On the right, the tangent line is steep, downward, corresponding to a small value of f'.

If a function is increasing and concave down, then its rate of increase is slowing; it is "leveling off." If the function is decreasing and concave down, then the *rate* of decrease is decreasing. The function is decreasing at a faster and faster rate.

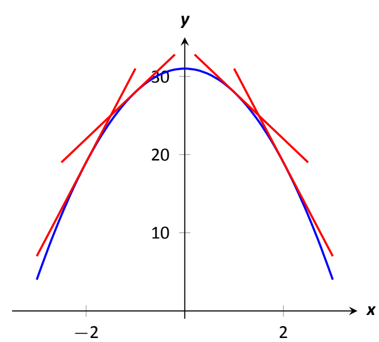


Figure 3.4.2: A function f with a concave down graph. Notice how the slopes of the tangent lines, when looking from left to right, are decreasing.

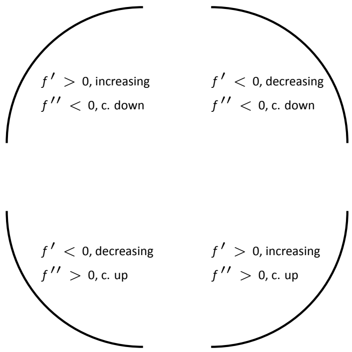


Figure 3.4.3: Demonstrating the 4 ways that concavity interacts with increasing/decreasing, along with the relationships with the first and second derivatives.

Our definition of concave up and concave down is given in terms of when the first derivative is increasing or decreasing. We can apply the results of the previous section and to find intervals on which a graph is concave up or down. That is, we recognize that f' is increasing when $f'' > 0$, etc.

Theorem 3.4.1 Test for Concavity

Let f be twice differentiable on an interval I. The graph of f is concave up if $f'' > 0$ on I, and is concave down if $f'' < 0$ on I.

If knowing where a graph is concave up/down is important, it makes sense that the places where the graph changes from one to the other is also important. This leads us to a definition.

Definition 3.4.2 Point of Inflection

A **point of inflection** is a point on the graph of f at which the concavity of f changes.

Note: Geometrically speaking, a function is concave up if its graph lies above its tangent lines. A function is concave down if its graph lies below its tangent lines.

Figure 3.4.4 shows a graph of a function with inflection points labeled.

If the concavity of f changes at a point $(c, f(c))$, then f' is changing from increasing to decreasing (or, decreasing to increasing) at $x = c$. That means that the sign of f'' is changing from positive to negative (or, negative to positive) at $x = c$. This leads to the following theorem.

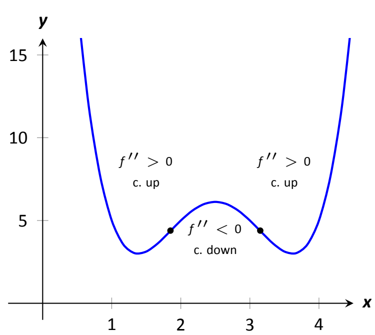


Figure 3.4.4: A graph of a function with its inflection points marked. The intervals where concave up/down are also indicated.

Theorem 3.4.2 Locating Theorem for Inflection Points

Points of inflection of a function f ***may*** occur ***only*** where

1. $f''(c) = 0$
2. $f''(c)$ *does not exist.*.

We have identified the concepts of concavity and points of inflection. It is now time to practice using these concepts; given a function, we should be able to find its points of inflection and identify intervals on which it is concave up or down. We do so in the following examples.

NOTE In Figure 3.4.5, f"(0) = 0, but x = 0 is not an inflection point because it does not connect concave up with concave down. Thus the ***may*** in the Locating Theorem.

RELATED NOTE The word 'inflection' literally means not bending. In old calculus textbooks, that was the meaning of inflection point. This idea is still relevant. Near a point where the second derivative is 0, as in Figure 3.4.5, a curve is straighter than the usual local linearity possessed by a differentiable curve.

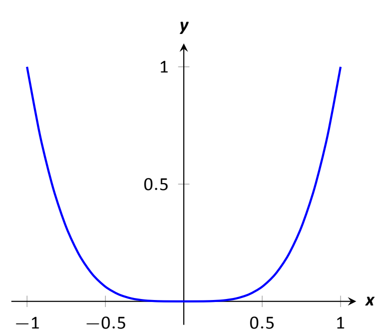
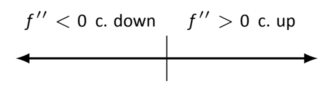


Figure 3.4.5: A graph of $f(x) = x^4$. Clearly f is always concave up, despite the fact that $f''(x) = 0$ when $x = 0$. It this example, the *possible* point of inflection $(0, 0)$ is not a point of inflection.

0

Example 3.4.1 Finding intervals of concave up/down, inflection points Let $f(x) = x^3 - 3x + 1$. Find the inflection points of f and the intervals on which it is concave up/down.

Figure 3.4.6: A number line determining the concavity of f in Example 3.4.1.

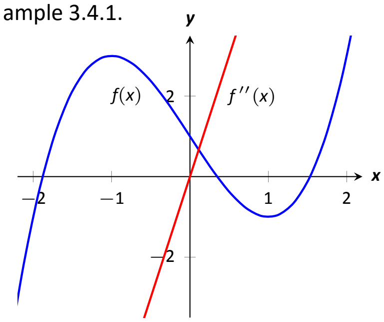


SOLUTION We start by finding $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. To find the inflection points, we use Theorem 3.4.2 and find where $f''(x) = 0$ or where f'' is undefined. We find f'' is always defined, and is 0 only when $x = 0$. So the point $(0, 1)$ is the only possible point of inflection.

This possible inflection point divides the real line into two intervals, $(-\infty, 0)$ and $(0, \infty)$. We use a process similar to the one used in the previous section to determine increasing/decreasing. Pick any $c < 0$; $f''(c) < 0$ so f is concave down on $(-\infty, 0)$. Pick any $c > 0$; $f''(c) > 0$ so f is concave up on $(0, \infty)$. Since the concavity changes at $x = 0$, the point $(0, 1)$ is an inflection point.

The number line in Figure 3.4.6 illustrates the process of determining concavity; Figure 3.4.7 shows a graph of f and f'', confirming our results. Notice how f is concave down precisely when $f''(x) < 0$ and concave up when $f''(x) > 0$.

Figure 3.4.7: A graph of $f(x)$ used in Example 3.4.1.



Example 3.4.2 Finding intervals of concave up/down, inflection points Let $f(x) = x/(x^2 - 1)$. Find the inflection points of f and the intervals on which it is concave up/down.

SOLUTION We need to find f' and f''. Using the Quotient Rule and simplifying, we find

$$f'(x) = \frac{-(1 + x^2)}{(x^2 - 1)^2} \quad \text{and} \quad f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

To find the possible points of inflection, we seek to find where $f''(x) = 0$ and where f'' is not defined. Solving $f''(x) = 0$ reduces to solving $2x(x^2 + 3) = 0$; we find $x = 0$. We find that f'' is not defined when $x = \pm 1$, for then the denominator of f'' is 0. We also note that f itself is not defined at $x = \pm 1$, having a domain of $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. Since the domain of f is the union of three intervals, it makes sense that the concavity of f could switch across intervals. We technically cannot say that f has a point of inflection at $x = \pm 1$ as they are not part of the domain, but we must still consider these x-values to be important and will include them in our number line.

The important x-values at which concavity might switch are $x = -1$, $x = 0$ and $x = 1$, which split the number line into four intervals as shown in Figure 3.4.7. We determine the concavity on each. Keep in mind that all we are concerned with is the *sign* of f'' on the interval.

Interval 1, $(-\infty, -1)$: Select a number c in this interval with a large magnitude (for instance, $c = -100$). The denominator of $f''(x)$ will be positive. In the numerator, the $(c^2 + 3)$ will be positive and the $2c$ term will be negative. Thus the numerator is negative and $f''(c)$ is negative. We conclude f is concave down on $(-\infty, -1)$.

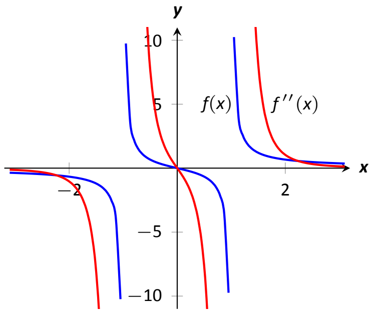


Figure 3.4.8: A graph of $f(x)$ and $f''(x)$ in Example 3.4.2.

Interval 2, $(-1, 0)$: For any number c in this interval, the term $2c$ in the numerator will be negative, the term $(c^2 + 3)$ in the numerator will be positive, and the term $(c^2 - 1)^3$ in the denominator will be negative. Thus $f''(c) > 0$ and f is concave up on this interval.

Interval 3, $(0, 1)$: Any number c in this interval will be positive and "small." Thus the numerator is positive while the denominator is negative. Thus $f''(c) < 0$ and f is concave down on this interval.

Interval 4, $(1, \infty)$: Choose a large value for c. It is evident that $f''(c) > 0$, so we conclude that f is concave up on $(1, \infty)$.

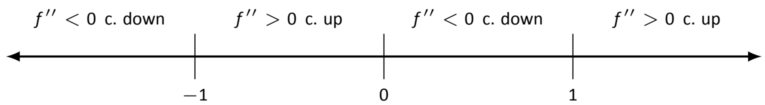


Figure 3.4.8: Number line for f in Example 3.4.2.

We conclude that f is concave up on $(-1, 0)$ and $(1, \infty)$ and concave down on $(-\infty, -1)$ and $(0, 1)$. There is only one point of inflection, $(0, 0)$, as f is not defined at $x = \pm 1$. Our work is confirmed by the graph of f in Figure 3.4.8. Notice how f is concave up whenever f'' is positive, and concave down when f'' is negative.

Recall that relative maxima and minima of f are found at critical points of f; that is, they are found when $f'(x) = 0$ or when f' is undefined. Likewise, the relative maxima and minima of f' are found when $f''(x) = 0$ or when f'' is undefined; note that these are the inflection points of f.

What does a "relative maximum of f'" *mean*? The derivative measures the rate of change of f; maximizing f' means finding where f is increasing the most – where f has the steepest tangent line. A similar statement can be made for minimizing f'; it corresponds to where f has the steepest negatively–sloped tangent line.

We utilize this concept in the next example.

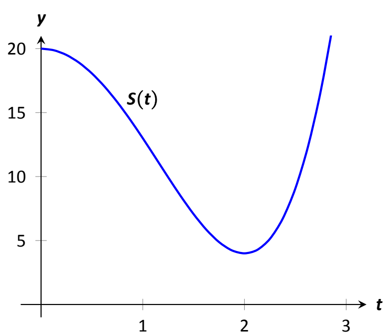


Figure 3.4.9: A graph of $S(t)$ in Example 3.4.3, modeling the sale of a product over time.

Example 3.4.3 Understanding inflection points

The sales of a certain product over a three-year span are modeled by $S(t) = t^4 - 8t^2 + 20$, where t is the time in years, shown in Figure 3.4.9. Over the first two years, sales are decreasing. Find the point at which sales are decreasing at their greatest rate.

Solution We want to maximize the rate of decrease, which is to say, we want to find where S' has a minimum. To do this, we find where S'' i s 0. We find $S'(t) = 4t^3 - 16t$ and $S''(t) = 12t^2 - 16$. Setting $S''(t) = 0$ and solving, we get $t = \sqrt{4/3} \doteq 1.16$ (we ignore the negative value of t since it does not lie in the domain of our function S).

This is both the inflection point and the point of maximum decrease. This is the point at which things first start looking up for the company. After the inflection point, it will still take some time before sales start to increase, but at least sales are not decreasing quite as quickly as they had been.

A graph of $S(t)$ and $S'(t)$ is given in Figure 3.4.10. When $S'(t) < 0$, sales are decreasing; note how at $t \doteq 1.16$, $S'(t)$ is minimized. That is, sales are decreasing at the fastest rate at $t \doteq 1.16$. On the interval of $(1.16, 2)$, S is decreasing but concave up, so the decline in sales is "leveling off."

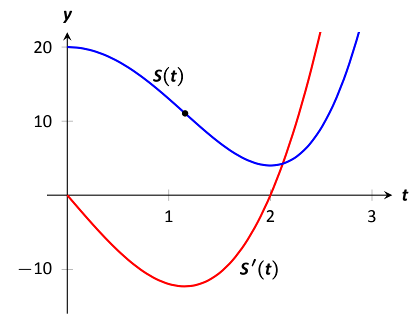


Figure 3.4.10: A graph of $S(t)$ in Example 3.4.3 along with $S'(t)$.

Not every critical point corresponds to a relative extrema; $f(x) = x^3$ has a critical point at $(0, 0)$ but no relative maximum or minimum. Likewise, just because $f''(x) = 0$ we cannot conclude concavity changes at that point. We were careful before to use terminology "*possible* point of inflection" since we needed to check to see if the concavity changed. The canonical example of $f''(x) = 0$ *without* concavity changing is $f(x) = x^4$. At $x = 0$, $f''(x) = 0$ but f is always concave up, as shown in Figure 3.4.11.

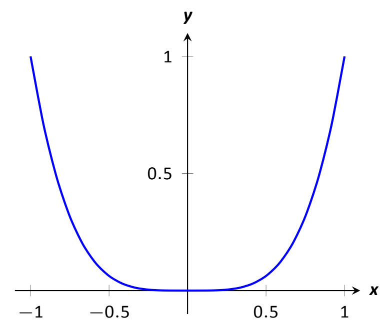


Figure 3.4.11: A graph of $f(x) = x^4$. Clearly f is always concave up, despite the fact that $f''(x) = 0$ when $x = 0$. It this example, the *possible* point of inflection $(0, 0)$ is not a point of inflection.

The Second Derivative Test

The first derivative of a function gave us a test to find if a critical value corresponded to a relative maximum, minimum, or neither. The second derivative gives us another way to test if a critical point is a local maximum or minimum. The following theorem officially states something that is intuitive: if a critical value occurs in a region where a function f is concave up, then that critical value must correspond to a relative minimum of f, etc. See Figure 3.4.12 for a visual-ization of this.

Theorem 3.4.3 The Second Derivative Test

Let c be a critical value of f where $f''(c)$ is defined.

1. If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.

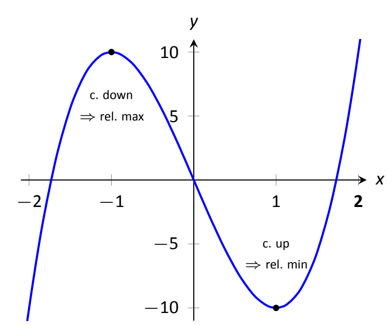


Figure 3.4.12: Demonstrating the fact that relative maxima occur when the graph is concave down and relative minima occur when the graph is concave up.

The Second Derivative Test relates to the First Derivative Test in the following way. If $f''(c) > 0$, then the graph is concave up at a critical point c and f' itself is growing. Since $f'(c) = 0$ and f' is growing at c, then it must go from negative to positive at c. This means the function goes from decreasing to increasing, indicating a local minimum at c.

**NOTE The Second Derivative Test is quite easy to apply.
It has been used in physics and other applications.**

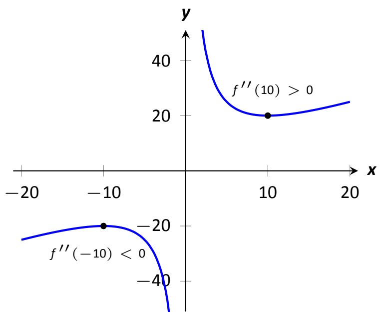


Figure 3.4.13: A graph of $f(x)$ in Example 3.4.4. The second derivative is evaluated at each critical point. When the graph is concave up, the critical point represents a local minimum; when the graph is concave down, the critical point represents a local maximum.

Example 3.4.4 Using the Second Derivative Test

Let $f(x) = 100/x + x$. Find the critical points of f and use the Second Derivative Test to label them as relative maxima or minima.

Solution We find $f'(x) = -100/x^2 + 1$ and $f''(x) = 200/x^3$. We set $f'(x) = 0$ and solve for x to find the critical values (note that f' is not defined at $x = 0$, but neither is f so this is not a critical value.) We find the critical values are $x = \pm 10$. Evaluating f'' at $x = 10$ gives $0.1 > 0$, so there is a local minimum at $x = 10$. Evaluating $f''(-10) = -0.1 < 0$, determining a relative maximum at $x = -10$. These results are confirmed in Figure 3.4.13.

We have been learning how the first and second derivatives of a function relate information about the graph of that function. We have found intervals of increasing and decreasing, intervals where the graph is concave up and down, along with the locations of relative extrema and inflection points. In Chapter 1 we saw how limits explained asymptotic behavior. In the next section we combine all of this information to produce accurate sketches of functions.

Exercises 3.4

Terms and Concepts

1. Sketch a graph of a function $f(x)$ that is concave up on $(0, 1)$ and is concave down on $(1, 2)$.

2. Sketch a graph of a function $f(x)$ that is:

   (a) Increasing, concave up on $(0, 1)$,

   (b) increasing, concave down on $(1, 2)$,

   (c) decreasing, concave down on $(2, 3)$ and

   (d) increasing, concave down on $(3, 4)$.

3. Is is possible for a function to be increasing and concave down on $(0, \infty)$ with a horizontal asymptote of $y = 1$? If so, give a sketch of such a function.

4. Is is possible for a function to be increasing and concave up on $(0, \infty)$ with a horizontal asymptote of $y = 1$? If so, give a sketch of such a function.

Problems

In Exercises 5 – 14, a function $f(x)$ is given.

(a) Compute $f''(x)$.

(b) Graph f and f'' on the same axes (using technology is permitted) and verify Theorem 3.4.1.

5. $f(x) = -7x + 3$

6. $f(x) = -4x^2 + 3x - 8$

7. $f(x) = 4x^2 + 3x - 8$

8. $f(x) = x^3 - 3x^2 + x - 1$

9. $f(x) = -x^3 + x^2 - 2x + 5$

10. $f(x) = \sin x$

11. $f(x) = \tan x$

12. $f(x) = \dfrac{1}{x^2 + 1}$

13. $f(x) = \dfrac{1}{x}$

14. $f(x) = \dfrac{1}{x^2}$

In Exercises 15 – 28, a function $f(x)$ is given.

(a) Find the possible points of inflection of f.

(b) Create a number line to determine the intervals on which f is concave up or concave down.

15. $f(x) = x^2 - 2x + 1$

16. $f(x) = -x^2 - 5x + 7$

17. $f(x) = x^3 - x + 1$

18. $f(x) = 2x^3 - 3x^2 + 9x + 5$

19. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$

20. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$

21. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

22. $f(x) = \sec x$ on $(-3\pi/2, 3\pi/2)$

23. $f(x) = \dfrac{1}{x^2 + 1}$

24. $f(x) = \dfrac{x}{x^2 - 1}$

25. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

26. $f(x) = x^2 e^x$

27. $f(x) = x^2 \ln x$

28. $f(x) = e^{-x^2}$

In Exercises 29 – 42, a function $f(x)$ is given. Find the critical points of f and use the Second Derivative Test, when possible, to determine the relative extrema. (Note: these are the same functions as in Exercises 15 – 28.)

29. $f(x) = x^2 - 2x + 1$

30. $f(x) = -x^2 - 5x + 7$

31. $f(x) = x^3 - x + 1$

32. $f(x) = 2x^3 - 3x^2 + 9x + 5$

33. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$

34. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$

35. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

36. $f(x) = \sec x$ on $(-3\pi/2, 3\pi/2)$

37. $f(x) = \dfrac{1}{x^2+1}$

38. $f(x) = \dfrac{x}{x^2-1}$

39. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

40. $f(x) = x^2 e^x$

41. $f(x) = x^2 \ln x$

42. $f(x) = e^{-x^2}$

In Exercises 43 – 56, a function $f(x)$ is given. Find the x values where $f'(x)$ has a local maximum or minimum. (Note: these are the same functions as in Exercises 15 – 28.)

43. $f(x) = x^2 - 2x + 1$

44. $f(x) = -x^2 - 5x + 7$

45. $f(x) = x^3 - x + 1$

46. $f(x) = 2x^3 - 3x^2 + 9x + 5$

47. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$

48. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$

49. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

50. $f(x) = \sec x$ on $(-3\pi/2, 3\pi/2)$

51. $f(x) = \dfrac{1}{x^2+1}$

52. $f(x) = \dfrac{x}{x^2-1}$

53. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

54. $f(x) = x^2 e^x$

55. $f(x) = x^2 \ln x$

56. $f(x) = e^{-x^2}$

57. Give a graphical of a function for which
    1. f''(c) = 0, no inflection point
    2. f''(c) DNE, no inflection point.

Solutions 3.4

1. Answers will vary.
2. Answers will vary.
3. Yes; Answers will vary.
4. No.
5. Graph and verify.
6. Graph and verify.
7. Graph and verify.
8. Graph and verify.
9. Graph and verify.
10. Graph and verify.
11. Graph and verify.
12. Graph and verify.
13. Graph and verify.
14. Graph and verify.
15. Possible points of inflection: none; concave up on $(-\infty, \infty)$
16. Possible points of inflection: none; concave down on $(-\infty, \infty)$
17. Possible points of inflection: $x = 0$; concave down on $(-\infty, 0)$; concave up on $(0, \infty)$
18. Possible points of inflection: $x = 1/2$; concave down on $(-\infty, 1/2)$; concave up on $(1/2, \infty)$
19. Possible points of inflection: $x = -2/3, 0$; concave down on $(-2/3, 0)$; concave up on $(-\infty, -2/3)$ and $(0, \infty)$
20. Possible points of inflection: $x = (1/3)(2 \pm \sqrt{7})$; concave up on $((1/3)(2-\sqrt{7}), (1/3)(2+\sqrt{7}))$; concave down on $(-\infty, (1/3)(2-\sqrt{7}))$ and $((1/3)(2+\sqrt{7}), \infty)$
21. Possible points of inflection: $x = 1$; concave up on $(-\infty, \infty)$
22. Possible points of inflection: $f''(x)$ is not defined (nor is f) at $x = -\pi/2, \pi/2$; concave down on $(-3\pi/2, -\pi/2)$ and $(\pi/2, 3\pi/2)$ concave up on $(-\pi/2, \pi/2)$
23. Possible points of inflection: $x = \pm 1/\sqrt{3}$; concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$; concave up on $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$
24. Possible points of inflection: $x = 0, \pm 1$; concave down on $(-\infty, -1)$ and $(0, 1)$ concave up on $(-1, 0)$ and $(1, \infty)$
25. Possible points of inflection: $x = -\pi/4, 3\pi/4$; concave down on $(-\pi/4, 3\pi/4)$ concave up on $(-\pi, -\pi/4)$ and $(3\pi/4, \pi)$
26. Possible points of inflection: $x = -2 \pm \sqrt{2}$; concave down on $(-2-\sqrt{2}, -2+\sqrt{2})$ concave up on $(-\infty, -2-\sqrt{2})$ and $(-2+\sqrt{2}, \infty)$
27. Possible points of inflection: $x = 1/e^{3/2}$; concave down on $(0, 1/e^{3/2})$ concave up on $(1/e^{3/2}, \infty)$
28. Possible points of inflection: $x = \pm 1/\sqrt{2}$; concave down on $(-1/\sqrt{2}, 1/\sqrt{2})$ concave up on $(-\infty, -1/\sqrt{2})$ and $(1/\sqrt{2}, \infty)$
29. min: $x = 1$
30. max: $x = -5/2$
31. max: $x = -1/\sqrt{3}$ min: $x = 1/\sqrt{3}$
32.
33. min: $x = 1$
34. max: $x = -1, 2$; min: $x = 1$
35. min: $x = 1$
36. max: at $x = \pm\pi$ min: at $x = 0$
37. max: $x = 0$
38. critical values: $x = -1, 1$; no max/min
39. max: $x = \pi/4$; min: $x = -3\pi/4$
40. max: $x = -2$; min: $x = 0$
41. min: $x = 1/\sqrt{e}$
42. max: $x = 0$
43. f' has no maximal or minimal value.
44. f' has no maximal or minimal value
45. f' has a minimal value at $x = 0$
46. f' has a minimal value at $x = 1/2$
47. Possible points of inflection: $x = -2/3, 0$; f' has a relative min at: $x = 0$; relative max at: $x = -2/3$
48. f' has a relative max at: $x = (1/3)(2+\sqrt{7})$ relative min at: $x = (1/3)(2-\sqrt{7})$
49. f' has no relative extrema
50. $f'(x)$ has no relative extrema
51. f' has a relative max at $x = -1/\sqrt{3}$; relative min at $x = 1/\sqrt{3}$
52. f' has a relative max at $x = 0$
53. f' has a relative min at $x = 3\pi/4$; relative max at $x = -\pi/4$
54. f' has a relative max at $x = -2-\sqrt{2}$; relative min at $x = -2+\sqrt{2}$
55. f' has a relative min at $x = 1/\sqrt{e^3} = e^{-3/2}$
56. f' has a relative max at $x = -1/\sqrt{2}$; a relative min at $x = 1/\sqrt{2}$

3.5 Curve Sketching

We have been learning how we can understand the behavior of a function based on its first and second derivatives. While we have been treating the properties of a function separately (increasing and decreasing, concave up and concave down, etc.), we combine them here to produce an accurate graph of the function without plotting lots of extraneous points.

Why bother? Graphing utilities are very accessible, whether on a computer, a hand–held calculator, or a smartphone. These resources are usually very fast and accurate. We will see that our method is not particularly fast – it will require time (but it is not *hard*). So again: why bother?

We are attempting to understand the behavior of a function f based on the information given by its derivatives. While all of a function's derivatives relay information about it, it turns out that "most" of the behavior we care about is explained by f' and f''. Understanding the interactions between the graph of f and f' and f'' is important. To gain this understanding, one might argue that all that is needed is to look at lots of graphs. This is true to a point, but is somewhat similar to stating that one understands how an engine works after looking only at pictures. It is true that the basic ideas will be conveyed, but "hands–on" access increases understanding.

The following Key Idea summarizes what we have learned so far that is applicable to sketching graphs of functions and gives a framework for putting that information together. It is followed by several examples.

Key Idea 3.5.1 Curve Sketching

To produce an accurate sketch a given function f, consider the following steps.

1. Find the domain of f. Generally, we assume that the domain is the entire real line then find restrictions, such as where a denominator is 0 or where negatives appear under the radical.
2. Find the critical values of f.
3. Find the possible points of inflection of f.
4. Find the location of any vertical asymptotes of f (usually done in conjunction with item 1 above).
5. Consider the limits $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ to determine the end behavior of the function.

(continued)

Key Idea 3.5.1 Curve Sketching – Continued

6. Create a number line that includes all critical points, possible points of inflection, and locations of vertical asymptotes. For each interval created, determine whether f is increasing or decreasing, concave up or down.

7. Evaluate f at each critical point and possible point of inflection. Plot these points on a set of axes. Connect these points with curves exhibiting the proper concavity. Sketch asymptotes and x and y intercepts where applicable.

Example 3.5.1 Curve sketching
Use Key Idea 3.5.1 to sketch $f(x) = 3x^3 - 10x^2 + 7x + 5$.

Solution We follow the steps outlined in the Key Idea.

1. The domain of f is the entire real line; there are no values x for which $f(x)$ is not defined.

2. Find the critical values of f. We compute $f'(x) = 9x^2 - 20x + 7$. Use the Quadratic Formula to find the roots of f':

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(9)(7)}}{2(9)} = \frac{1}{9}\left(10 \pm \sqrt{37}\right) \Rightarrow x \doteq 0.435, 1.787.$$

3. Find the possible points of inflection of f. Compute $f''(x) = 18x - 20$. We have

$$f''(x) = 0 \Rightarrow x = 10/9 \doteq 1.111.$$

4. There are no vertical asymptotes.

5. We determine the end behavior using limits as x approaches $\pm$infinity.

$$\lim_{x\to-\infty} f(x) = -\infty \qquad \lim_{x\to\infty} f(x) = \infty.$$

We do not have any horizontal asymptotes.

6. We place the values $x = (10 \pm \sqrt{37})/9$ and $x = 10/9$ on a number line, as shown in Figure 3.5.1. We mark each subinterval as increasing or

N.B. Short Version
If you need a high quality graph, use a computer graphing utility. For a hand drawn graph:

Locate the local extreme points with a short tangent line
Locate possible points of inflection
Determine the behaviors at infinity:
vertical asymptotes
other asumptotes
Sketch the curve.

If in doubt, plot a few test points.

decreasing, concave up or down, using the techniques used in Sections 3.3 and 3.4.

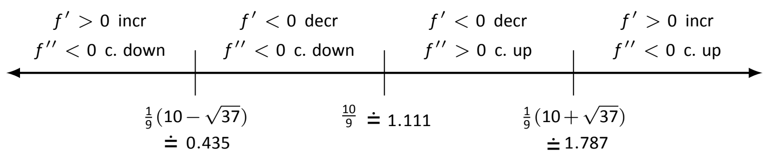


Figure 3.5.1: Number line for f in Example 3.5.1.

7. We plot the appropriate points on axes as shown in Figure 3.5.2(a) and connect the points with straight lines. In Figure 3.5.2(b) we adjust these lines to demonstrate the proper concavity. Our curve crosses the y axis at $y = 5$ and crosses the x axis near $x = -0.424$. In Figure 3.5.2(c) we show a graph of f drawn with a computer program, verifying the accuracy of our sketch.

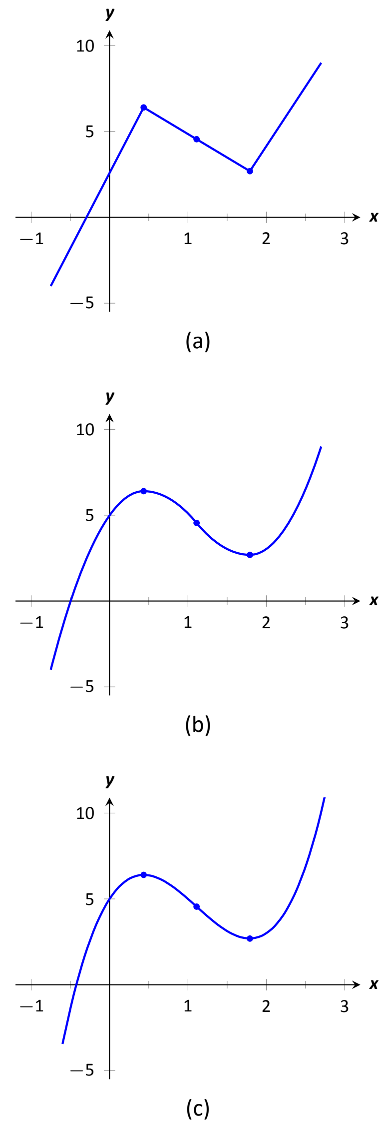


Figure 3.5.2: Sketching f in Example 3.5.1.

Example 3.5.2 Curve sketching

Sketch $f(x) = \dfrac{x^2 - x - 2}{x^2 - x - 6}$.

SOLUTION We again follow the steps outlined in Key Idea 3.5.1.

1. In determining the domain, we assume it is all real numbers and look for restrictions. We find that at $x = -2$ and $x = 3$, $f(x)$ is not defined. So the domain of f is $D = \{\text{real numbers } x \mid x \neq -2, 3\}$.

2. To find the critical values of f, we first find $f'(x)$. Using the Quotient Rule, we find
$$f'(x) = \frac{-8x+4}{(x^2+x-6)^2} = \frac{-8x+4}{(x-3)^2(x+2)^2}.$$
$f'(x) = 0$ when $x = 1/2$, and f' is undefined when $x = -2, 3$. Since f' is undefined only when f is, these are not critical values. The only critical value is $x = 1/2$.

3. To find the possible points of inflection, we find $f''(x)$, again employing the Quotient Rule:
$$f''(x) = \frac{24x^2 - 24x + 56}{(x-3)^3(x+2)^3}.$$
We find that $f''(x)$ is never 0 (setting the numerator equal to 0 and solving for x, we find the only roots to this quadratic are imaginary) and f'' is

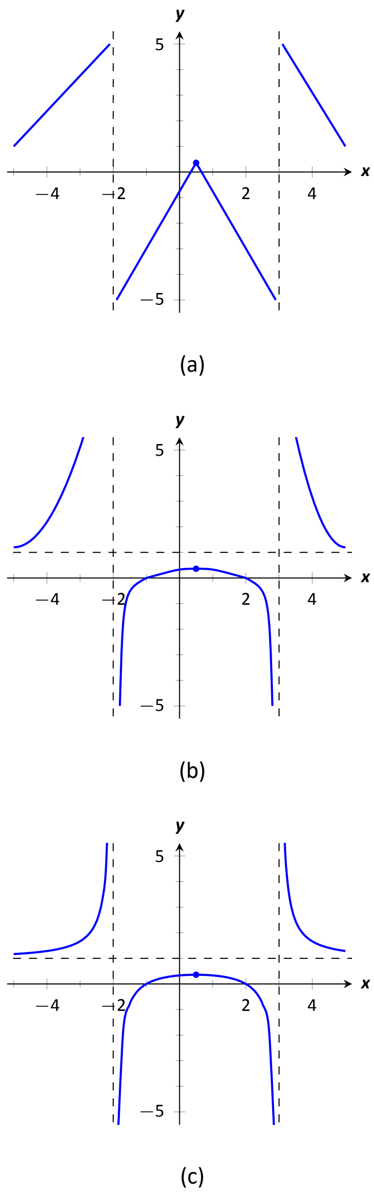


Figure 3.5.4: Sketching f in Example 3.5.2.

undefined when $x = -2, 3$. Thus concavity will possibly only change at $x = -2$ and $x = 3$.

4. The vertical asymptotes of f are at $x = -2$ and $x = 3$, the places where f is undefined.

5. There is a horizontal asymptote of $y = 1$, as $\lim_{x\to-\infty} f(x) = 1$ and $\lim_{x\to\infty} f(x) = 1$.

6. We place the values $x = 1/2$, $x = -2$ and $x = 3$ on a number line as shown in Figure 3.5.3. We mark in each interval whether f is increasing or decreasing, concave up or down. We see that f has a relative maximum at $x = 1/2$; concavity changes only at the vertical asymptotes.

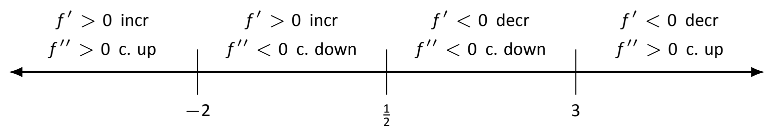


Figure 3.5.3: Number line for f in Example 3.5.2.

7. In Figure 3.5.4(a), we plot the points from the number line on a set of axes and connect the points with straight lines to get a general idea of what the function looks like (these lines effectively only convey increasing/decreasing information). In Figure 3.5.4(b), we adjust the graph with the appropriate concavity. We also show f crossing the x axis at $x = -1$ and $x = 2$.

Figure 3.5.4(c) shows a computer generated graph of f, which verifies the accuracy of our sketch.

Example 3.5.3 Curve sketching

Sketch $f(x) = \dfrac{5(x-2)(x+1)}{x^2+2x+4}$.

Solution We again follow Key Idea 3.5.1.

1. We assume that the domain of f is all real numbers and consider restrictions. The only restrictions come when the denominator is 0, but this never occurs. Therefore the domain of f is all real numbers, $\mathbb{R}$.

2. We find the critical values of f by setting $f'(x) = 0$ and solving for x. We find

$$f'(x) = \frac{15x(x+4)}{(x^2+2x+4)^2} \quad \Rightarrow \quad f'(x) = 0 \text{ when } x = -4, 0.$$

3. We find the possible points of inflection by solving $f''(x) = 0$ for x. We find
$$f''(x) = -\frac{30x^3 + 180x^2 - 240}{(x^2 + 2x + 4)^3}.$$
The cubic in the numerator does not factor very "nicely." We instead approximate the roots at $x = -5.759$, $x = -1.305$ and $x = 1.064$.

4. There are no vertical asymptotes.

5. We have a horizontal asymptote of $y = 5$, as $\lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to\infty} f(x) = 5$.

6. We place the critical points and possible points on a number line as shown in Figure 3.5.5 and mark each interval as increasing/decreasing, concave up/down appropriately.

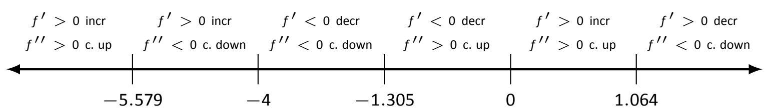


Figure 3.5.5: Number line for f in Example 3.5.3.

7. In Figure 3.5.6(a) we plot the significant points from the number line as well as the two roots of f, $x = -1$ and $x = 2$, and connect the points with straight lines to get a general impression about the graph. In Figure 3.5.6(b), we add concavity. Figure 3.5.6(c) shows a computer generated graph of f, affirming our results.

In each of our examples, we found a few, significant points on the graph of f that corresponded to changes in increasing/decreasing or concavity. We connected these points with straight lines, then adjusted for concavity, and finished by showing a very accurate, computer generated graph.

Why are computer graphics so good? It is not because computers are "smarter" than we are. Rather, it is largely because computers are much faster at computing than we are. In general, computers graph functions much like most students do when first learning to draw graphs: they plot equally spaced points, then connect the dots using lines. By using lots of points, the connecting lines are short and the graph looks smooth.

This does a fine job of graphing in most cases (in fact, this is the method used for many graphs in this text). However, in regions where the graph is very "curvy," this can generate noticeable sharp edges on the graph unless a large number of points are used. High quality computer algebra systems, such as

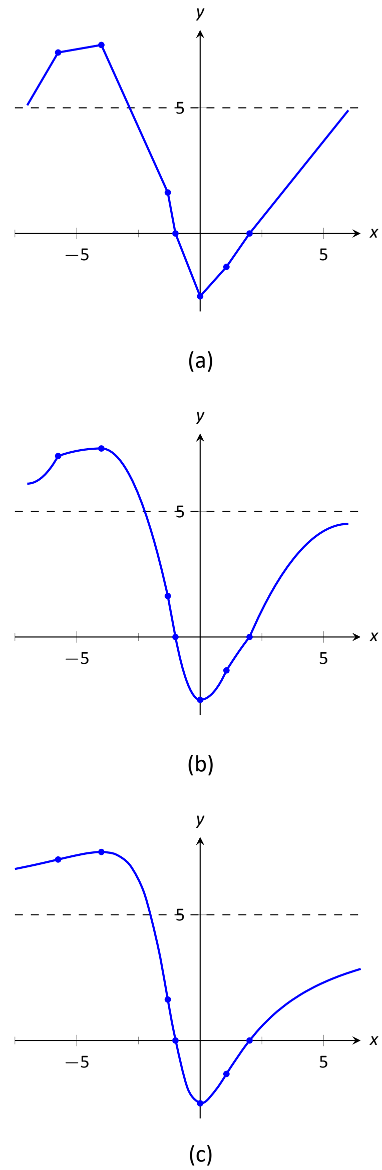


Figure 3.5.6: Sketching f in Example 3.5.3.

Mathematica, use special algorithms to plot lots of points only where the graph is "curvy."

In Figure 3.5.7, a graph of $y = \sin x$ is given, generated by *Mathematica*. The small points represent each of the places *Mathematica* sampled the function. Notice how at the "bends" of $\sin x$, lots of points are used; where $\sin x$ is relatively straight, fewer points are used. (Many points are also used at the endpoints to ensure the "end behavior" is accurate.) In fact, in the interval of length 0.2 centered around $\pi/2$, *Mathematica* plots 72 of the 431 points plotted; that is, it plots about 17% of its points in a subinterval that accounts for about 3% of the total interval length.

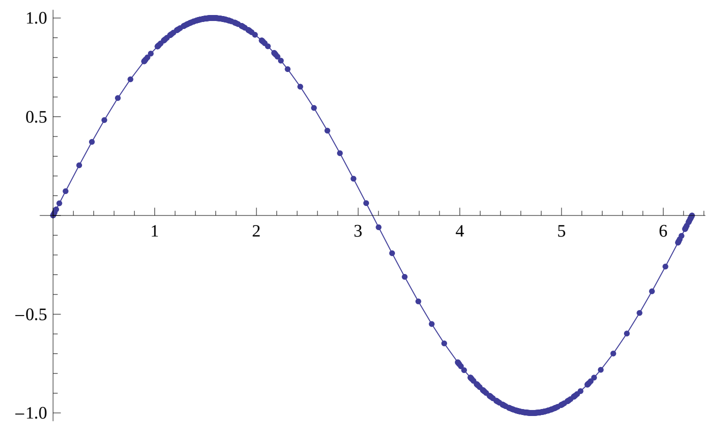


Figure 3.5.7: A graph of $y = \sin x$ generated by *Mathematica*.

How does *Mathematica* know where the graph is "curvy"? Calculus. When we study *curvature* in a later chapter, we will see how the first and second derivatives of a function work together to provide a measurement of "curviness." *Mathematica* employs algorithms to determine regions of "high curvature" and plots extra points there.

Again, the goal of this section is not "How to graph a function when there is no computer to help." Rather, the goal is "Understand that the shape of the graph of a function is largely determined by understanding the behavior of the function at a few key places." In Example 3.5.3, we were able to accurately sketch a complicated graph using only 5 points and knowledge of asymptotes!

There are many applications of our understanding of derivatives beyond curve sketching. The next chapter explores some of these applications, demonstrating just a few kinds of problems that can be solved with a basic knowledge of differentiation.

Exercises 3.5

Terms and Concepts

1. Why is sketching curves by hand beneficial even though technology is ubiquitous?

2. What does "ubiquitous" mean?

3. T/F: When sketching graphs of functions, it is useful to find the critical points.

4. T/F: When sketching graphs of functions, it is useful to find the possible points of inflection.

5. T/F: When sketching graphs of functions, it is useful to find the horizontal and vertical asymptotes.

6. T/F: When sketching graphs of functions, one need not plot any points at all.

Problems

In Exercises 7 – 12, practice using Key Idea 3.5.1 by applying the principles to the given functions with familiar graphs.

7. $f(x) = 2x + 4$

8. $f(x) = -x^2 + 1$

9. $f(x) = \sin x$

10. $f(x) = e^x$

11. $f(x) = \dfrac{1}{x}$

12. $f(x) = \dfrac{1}{x^2}$

In Exercises 13 – 26, sketch a graph of the given function using Key Idea 3.5.1. Show all work; check your answer with technology.

13. $f(x) = x^3 - 2x^2 + 4x + 1$

14. $f(x) = -x^3 + 5x^2 - 3x + 2$

15. $f(x) = x^3 + 3x^2 + 3x + 1$

16. $f(x) = x^3 - x^2 - x + 1$

17. $f(x) = (x-2)\ln(x-2)$

18. $f(x) = (x-2)^2\ln(x-2)$

19. $f(x) = \dfrac{x^2-4}{x^2}$

20. $f(x) = \dfrac{x^2-4x+3}{x^2-6x+8}$

21. $f(x) = \dfrac{x^2-2x+1}{x^2-6x+8}$

22. $f(x) = x\sqrt{x+1}$

23. $f(x) = x^2e^x$

24. $f(x) = \sin x \cos x$ on $[-\pi, \pi]$

25. $f(x) = (x-3)^{2/3} + 2$

26. $f(x) = \dfrac{(x-1)^{2/3}}{x}$

In Exercises 27 – 30, a function with the parameters a and b are given. Describe the critical points and possible points of inflection of f in terms of a and b.

27. $f(x) = \dfrac{a}{x^2+b^2}$

28. $f(x) = ax^2 + bx + 1$

29. $f(x) = \sin(ax+b)$

30. $f(x) = (x-a)(x-b)$

31. Given $x^2 + y^2 = 1$, use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Use this information to justify the sketch of the unit circle.

Solutions 3.5

1. Answers will vary.
2. Found everywhere.
3. T
4. T
5. T
6. F
7. A good sketch will include the x and y intercepts and draw the appropriate line.
8. A good sketch will include the x and y intercepts..
9. Use technology to verify sketch.
10. Use technology to verify sketch.
11. Use technology to verify sketch.
12. Use technology to verify sketch.
13. Use technology to verify sketch.
14. Use technology to verify sketch.
15. Use technology to verify sketch.
16. Use technology to verify sketch.
17. Use technology to verify sketch.
18. Use technology to verify sketch.
19. Use technology to verify sketch.
20. Use technology to verify sketch.
21. Use technology to verify sketch.
22. Use technology to verify sketch.
23. Use technology to verify sketch.
24. Use technology to verify sketch.
25. Use technology to verify sketch.
26. Use technology to verify sketch.
27. Critical point: $x = 0$ Points of inflection: $\pm b/\sqrt{3}$
28. Critical point: $x = -b/(2a)$ No points of inflection
29. Critical points: $x = \frac{n\pi/2-b}{a}$, where n is an odd integer Points of inflection: $(n\pi - b)/a$, where n is an integer.
30. Critical point: $x = (a+b)/2$ Points of inflection: none
31. $\frac{dy}{dx} = -x/y$, so the function is increasing in second and fourth quadrants, decreasing in the first and third quadrants.

    $\frac{d^2y}{dx^2} = -1/y - x^2/y^3$, which is positive when $y < 0$ and is negative when $y > 0$. Hence the function is concave down in the first and second quadrants and concave up in the third and fourth quadrants.

4: Applications of the Derivative

In Chapter 3, we learned how the first and second derivatives of a function influence its graph. In this chapter we explore other applications of the derivative.

4.1 Newton's Method

Solving equations is one of the most important things we do in mathematics, yet we are surprisingly limited in what we can solve analytically. For instance, equations as simple as $x^5+x+1=0$ or $\cos x = x$ cannot be solved by algebraic methods in terms of familiar functions. Fortunately, there are methods that can give us *approximate* solutions to equations like these. These methods can usually give an approximation correct to as many decimal places as we like. In Section 1.5 we learned about the Bisection Method. This section focuses on another technique (which generally works faster), called Newton's Method.

Newton's Method is built around tangent lines. The main idea is that if x is sufficiently close to a root of $f(x)$, then the tangent line to the graph at $(x, f(x))$ will cross the x-axis at a point closer to the root than x.

We start Newton's Method with an initial guess about roughly where the root is. Call this x_0. (See Figure 4.1.1(a).) Draw the tangent line to the graph at $(x_0, f(x_0))$ and see where it meets the x-axis. Call this point x_1. Then repeat the process – draw the tangent line to the graph at $(x_1, f(x_1))$ and see where it meets the x-axis. (See Figure 4.1.1(b).) Call this point x_2. Repeat the process again to get x_3, x_4, etc. This sequence of points will often converge rather quickly to a root of f.

We can use this *geometric* process to create an *algebraic* process. Let's look at how we found x_1. We started with the tangent line to the graph at $(x_0, f(x_0))$. The slope of this tangent line is $f'(x_0)$ and the equation of the line is

$$y = f'(x_0)(x-x_0)+f(x_0).$$

This line crosses the x-axis when $y=0$, and the x–value where it crosses is what we called x_1. So let $y=0$ and replace x with x_1, giving the equation:

$$0 = f'(x_0)(x_1-x_0)+f(x_0).$$

Now solve for x_1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

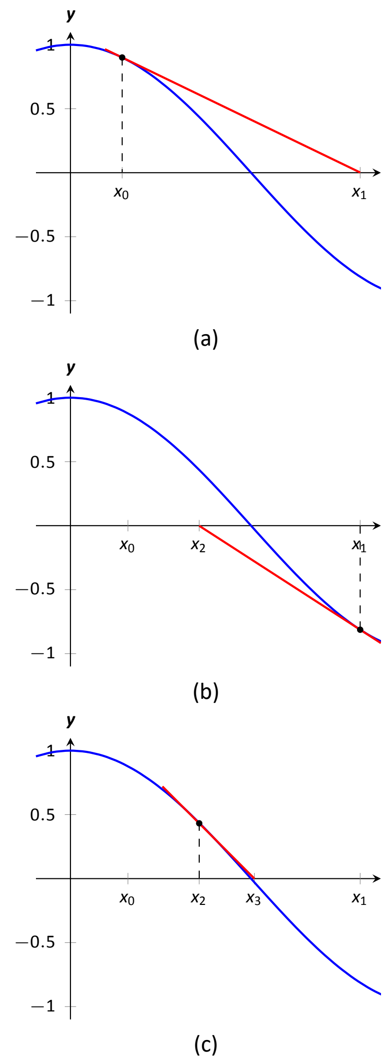


Figure 4.1.1: Demonstrating the geometric concept behind Newton's Method. Note how x_3 is very close to a solution to $f(x)=0$.

Since we repeat the same geometric process to find x_2 from x_1, we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, given an approximation x_n, we can find the next approximation, x_{n+1} as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We summarize this process as follows.

Key Idea 4.1.1 Newton's Method

Let f be a differentiable function on an interval I with a root in I. To approximate the value of the root, accurate to d decimal places:

1. Choose a value x_0 as an initial approximation of the root. (This is often done by looking at a graph of f.)
2. Create successive approximations iteratively; given an approximation x_n, compute the next approximation x_{n+1} as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. Stop the iterations when successive approximations do not differ in the first d places after the decimal point.

Note: Newton's Method is not infallible. The sequence of approximate values may not converge, or it may converge so slowly that one is "tricked" into thinking a certain approximation is better than it actually is. These issues will be discussed at the end of the section.

Let's practice Newton's Method with a concrete example.

Example 4.1.1 Using Newton's Method

Approximate the real root of $x^3 - x^2 - 1 = 0$, accurate to the first 3 places after the decimal, using Newton's Method and an initial approximation of $x_0 = 1$.

Solution To begin, we compute $f'(x) = 3x^2 - 2x$. Then we apply the

Newton's Method algorithm, outlined in Key Idea 4.1.1.

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1^3 - 1^2 - 1}{3 \cdot 1^2 - 2 \cdot 1} = 2,$$
$$x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{2^3 - 2^2 - 1}{3 \cdot 2^2 - 2 \cdot 2} = 1.625,$$
$$x_3 = 1.625 - \frac{f(1.625)}{f'(1.625)} = 1.625 - \frac{1.625^3 - 1.625^2 - 1}{3 \cdot 1.625^2 - 2 \cdot 1.625} \doteq 1.48579.$$
$$x_4 = 1.48579 - \frac{f(1.48579)}{f'(1.48579)} \doteq 1.46596$$
$$x_5 = 1.46596 - \frac{f(1.46596)}{f'(1.46596)} \doteq 1.46557$$

We performed 5 iterations of Newton's Method to find a root accurate to the first 3 places after the decimal; our final approximation is 1.465. The exact value of the root, to six decimal places, is 1.465571; It turns out that our x_5 is accurate to more than just 3 decimal places.

A graph of $f(x)$ is given in Figure 4.1.2. We can see from the graph that our initial approximation of $x_0 = 1$ was not particularly accurate; a closer guess would have been $x_0 = 1.5$. Our choice was based on ease of initial calculation, and shows that Newton's Method can be robust enough that we do not have to make a very accurate initial approximation.

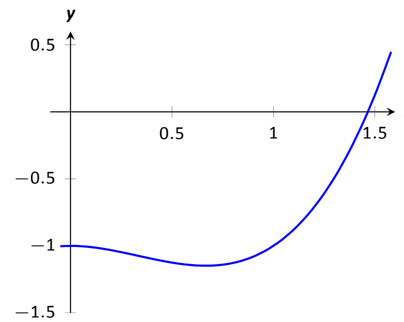


Figure 4.1.2: A graph of $f(x) = x^3 - x^2 - 1$ in Example 4.1.1.

We can automate this process on a calculator that has an `Ans` key that returns the result of the previous calculation. Start by pressing `1` and then `Enter`. (We have just entered our initial guess, $x_0 = 1$.) Now compute

$$\texttt{Ans} - \frac{f(\texttt{Ans})}{f'(\texttt{Ans})}$$

by entering the following and repeatedly press the `Enter` key:

```
Ans-(Ans^3-Ans^2-1)/(3*Ans^2-2*Ans)
```

Each time we press the `Enter` key, we are finding the successive approximations, x_1, x_2, …, and each one is getting closer to the root. In fact, once we get past around x_7 or so, the approximations don't appear to be changing. They actually are changing, but the change is far enough to the right of the decimal point that it doesn't show up on the calculator's display. When this happens, we can be pretty confident that we have found an accurate approximation.

Using a calculator in this manner makes the calculations simple; many iterations can be computed very quickly.

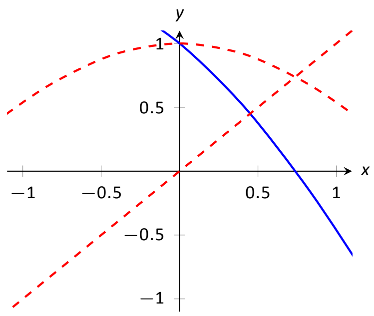


Figure 4.1.3: A graph of $f(x) = \cos x - x$ used to find an initial approximation of its root.

Example 4.1.2 Using Newton's Method to find where functions intersect

Use Newton's Method to approximate a solution to $\cos x = x$, accurate to 5 places after the decimal.

Solution Newton's Method provides a method of solving $f(x) = 0$; it is not (directly) a method for solving equations like $f(x) = g(x)$. However, this is not a problem; we can rewrite the latter equation as $f(x) - g(x) = 0$ and then use Newton's Method.

So we rewrite $\cos x = x$ as $\cos x - x = 0$. Written this way, we are finding a root of $f(x) = \cos x - x$. We compute $f'(x) = -\sin x - 1$. Next we need a starting value, x_0. Consider Figure 4.1.3, where $f(x) = \cos x - x$ is graphed. It seems that $x_0 = 0.75$ is pretty close to the root, so we will use that as our x_0. (The figure also shows the graphs of $y = \cos x$ and $y = x$, drawn with dashed lines. Note how they intersect at the same x value as when $f(x) = 0$.)

We now compute x_1, x_2, etc. The formula for x_1 is

$$x_1 = 0.75 - \frac{\cos(0.75) - 0.75}{-\sin(0.75) - 1} \doteq 0.7391111388.$$

Apply Newton's Method again to find x_2:

$$x_2 = 0.7391111388 - \frac{\cos(0.7391111388) - 0.7391111388}{-\sin(0.7391111388) - 1} \doteq 0.7390851334.$$

We can continue this way, but it is really best to automate this process. On a calculator with an Ans key, we would start by pressing 0.75, then `Enter`, inputting our initial approximation. We then enter:

```
Ans - (cos(Ans)-Ans)/(-sin(Ans)-1).
```

Repeatedly pressing the `Enter` key gives successive approximations. We quickly find:

$$x_3 = 0.7390851332$$
$$x_4 = 0.7390851332.$$

Our approximations x_2 and x_3 did not differ for at least the first 5 places after the decimal, so we could have stopped. However, using our calculator in the man-ner described is easy, so finding x_4 was not hard. It is interesting to see how we found an approximation, accurate to as many decimal places as our calculator displays, in just 4 iterations.

Read and understand only. The following material discusses how Newton's Method can fail.

Convergence of Newton's Method

What should one use for the initial guess, x_0? Generally, the closer to the actual root the initial guess is, the better. However, some initial guesses should be avoided. For instance, consider Example 4.1.1 where we sought the root to $f(x) = x^3 - x^2 - 1$. Choosing $x_0 = 0$ would have been a particularly poor choice. Consider Figure 4.1.4, where $f(x)$ is graphed along with its tangent line at $x = 0$. Since $f'(0) = 0$, the tangent line is horizontal and does not intersect the x–axis. Graphically, we see that Newton's Method fails.

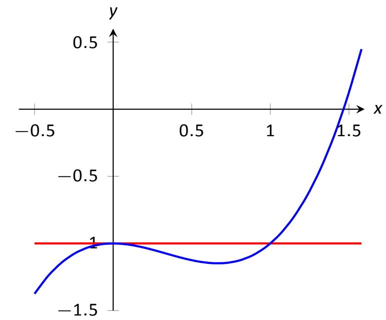


Figure 4.1.4: A graph of $f(x) = x^3 - x^2 - 1$, showing why an initial approximation of $x_0 = 0$ with Newton's Method fails.

We can also see analytically that it fails. Since

$$x_1 = 0 - \frac{f(0)}{f'(0)}$$

and $f'(0) = 0$, we see that x_1 is not well defined.

This problem can also occur if, for instance, it turns out that $f'(x_5) = 0$. Adjusting the initial approximation x_0 by a very small amount will likely fix the problem.

It is also possible for Newton's Method to not converge while each successive approximation is well defined. Consider $f(x) = x^{1/3}$, as shown in Figure 4.1.5. It is clear that the root is $x = 0$, but let's approximate this with $x_0 = 0.1$. Figure 4.1.5(a) shows graphically the calculation of x_1; notice how it is farther from the root than x_0. Figures 4.1.5(b) and (c) show the calculation of x_2 and x_3, which are even farther away; our successive approximations are getting worse. (It turns out that in this particular example, each successive approximation is twice as far from the true answer as the previous approximation.)

There is no "fix" to this problem; Newton's Method simply will not work and another method must be used.

While Newton's Method does not always work, it does work "most of the time," and it is generally very fast. Once the approximations get close to the root, Newton's Method can as much as double the number of correct decimal places with each successive approximation. A course in Numerical Analysis will introduce the reader to more iterative root finding methods, as well as give greater detail about the strengths and weaknesses of Newton's Method.

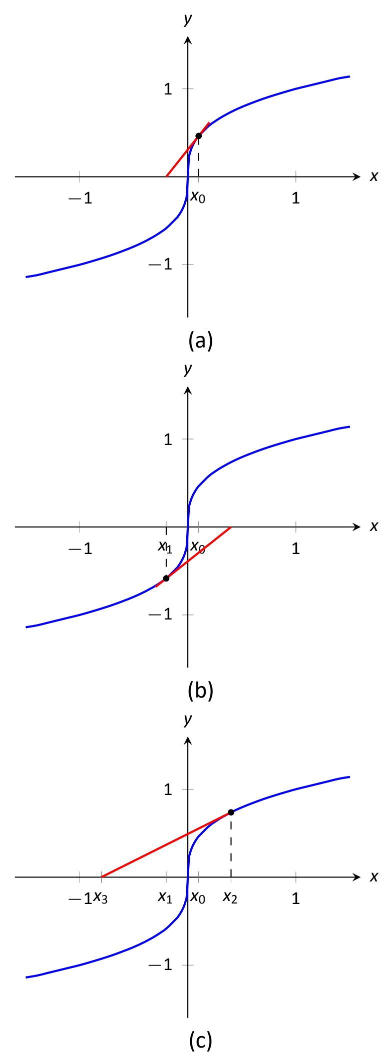


Figure 4.1.5: Newton's Method fails to find a root of $f(x) = x^{1/3}$, regardless of the choice of x_0.

Note In the modern world, one would normally use a scientific calculator or a computer algebraic systems to solve equations. Life is too short or time too valuable to do otherwise.

However, it is good to understand how the equation solver on your calculator or computer works. If you are writing a computer program to solve another problem and need to solve equations, you might wish to write a routine including Newton's Method.

Exercises 4.1

Terms and Concepts

1. T/F: Given a function $f(x)$, Newton's Method produces an exact solution to $f(x) = 0$.

2. T/F: In order to get a solution to $f(x) = 0$ accurate to d places after the decimal, at least $d + 1$ iterations of Newtons' Method must be used.

Problems

In Exercises 3 – 8, the roots of $f(x)$ are known or are easily found. Use 5 iterations of Newton's Method with the given initial approximation to approximate the root. Compare it to the known value of the root.

3. $f(x) = \cos x$, $x_0 = 1.5$

4. $f(x) = \sin x$, $x_0 = 1$

5. $f(x) = x^2 + x - 2$, $x_0 = 0$

6. $f(x) = x^2 - 2$, $x_0 = 1.5$

7. $f(x) = \ln x$, $x_0 = 2$

8. $f(x) = x^3 - x^2 + x - 1$, $x_0 = 1$

In Exercises 9 – 12, use Newton's Method to approximate all roots of the given functions accurate to 3 places after the decimal. If an interval is given, find only the roots that lie in that interval. Use technology to obtain good initial approximations.

9. $f(x) = x^3 + 5x^2 - x - 1$

10. $f(x) = x^4 + 2x^3 - 7x^2 - x + 5$

11. $f(x) = x^{17} - 2x^{13} - 10x^8 + 10$ on $(-2, 2)$

12. $f(x) = x^2 \cos x + (x - 1) \sin x$ on $(-3, 3)$

In Exercises 13 – 16, use Newton's Method to approximate when the given functions are equal, accurate to 3 places after the decimal. Use technology to obtain good initial approximations.

13. $f(x) = x^2$, $g(x) = \cos x$

14. $f(x) = x^2 - 1$, $g(x) = \sin x$

15. $f(x) = e^{x^2}$, $g(x) = \cos x$

16. $f(x) = x$, $g(x) = \tan x$ on $[-6, 6]$

17. Why does Newton's Method fail in finding a root of $f(x) = x^3 - 3x^2 + x + 3$ when $x_0 = 1$?

18. Why does Newton's Method fail in finding a root of $f(x) = -17x^4 + 130x^3 - 301x^2 + 156x + 156$ when $x_0 = 1$?

Solutions 4.1

1. F
2. F
3. $x_0 = 1.5$, $x_1 = 1.5709148$, $x_2 = 1.57018$. The approxima ons alternate between x = 1, x = 2 and x = 3.7963, $x_3 = 1.5707963$, $x_4 = 1.5707963$, $x_5 = 1.5707963$
4. $x_0 = 1$, $x_1 = -0.55740772$, $x_2 = 0.065936452$, $x_3 = -0.000095721919$, $x_4 = 2.9235662 * 10^{-13}$, $x_5 = 0$
5. $x_0 = 0$, $x_1 = 2$, $x_2 = 1.2$, $x_3 = 1.0117647$, $x_4 = 1.0000458$, $x_5 = 1$
6. $x_0 = 1.5$, $x_1 = 1.4166667$, $x_2 = 1.4142157$, $x_3 = 1.4142136$, $x_4 = 1.4142136$, $x_5 = 1.4142136$
7. $x_0 = 2$, $x_1 = 0.6137056389$, $x_2 = 0.9133412072$, $x_3 = 0.9961317034$, $x_4 = 0.9999925085$, $x_5 = 1$
8. $x_0 = 1$, $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_5 = 1$
9. roots are: $x = -5.156$, $x = -0.369$ and $x = 0.525$
10. roots are: $x = -3.714$, $x = -0.857$, $x = 1$ and $x = 1.571$
11. roots are: $x = -1.013$, $x = 0.988$, and $x = 1.393$
12. roots are: $x = -2.165$, $x = 0$, $x = 0.525$ and $x = 1.813$
13. $x = \pm 0.824$,
14. $x = -0.637$, $x = 1.410$
15. $x = \pm 0.743$
16. $x = \pm 4.493$, $x = 0$
17. The approximations alternate between $x = 1$ and $x = 2$.
18. The approximations alternate between $x = 1$, $x = 2$ and $x = 3$.

Asymptotic Equality

It's time to review the new 'near equality' that will be be useful in doing theory and applications in the next few chapters and sustain good calculus style later in your work life. It applies to infinitesimal, finite hyperreal, and infinite number calculations.

Definition A is ***asymptotically equal*** to B written $A \approx B$ means $\frac{A}{B} = 1 + \epsilon$ where ϵ is an infinitesimal.

Properties (proofs left as easy exercises)

1. $A \approx A$
2. $A \approx B \iff B \approx A$
3. $A \approx B, B \approx C \iff A \approx C$

NOTE again that $\approx$ is an excellent hyperreal approximation but yields an *exact* extended real; it's really a '*real* ='!

Theorem $a \approx A,\ b \approx B \iff a \cdot A \approx b \cdot B$

Theorem $a \approx A,\ b \approx B \iff \frac{a}{A} \approx \frac{b}{B}$

Note: $A \approx 0$ is never true. This will never be a serious problem in calculus. We can ignore this case there because the final answer will never be affected.

Examples in detail. Examine each graph carefully. Understand.

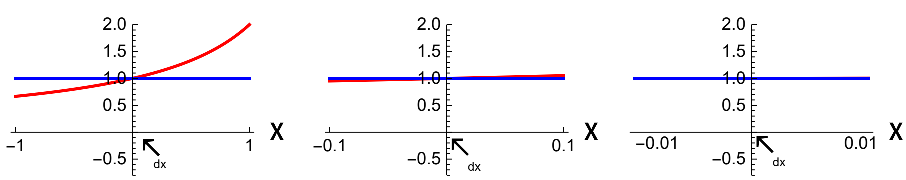
Infinitesimal Case $2\,dx - dx^2 \approx 2\,dx$

Proof $\frac{2\,dx - dx^2}{2\,dx} = 1 - \frac{dx}{2} = 1 + \epsilon$ where ϵ is an infinitesimal.

Next we illustrate the above with approximations of dx by 'small' real numbers.

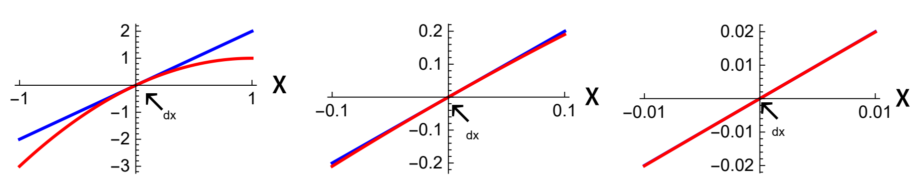
Graphically, compare the ratio with 1:

$\frac{2\,dx - dx^2}{2\,dx}$, 1



Graphically, compare individually:

$2dx - dx^2$, $2dx$

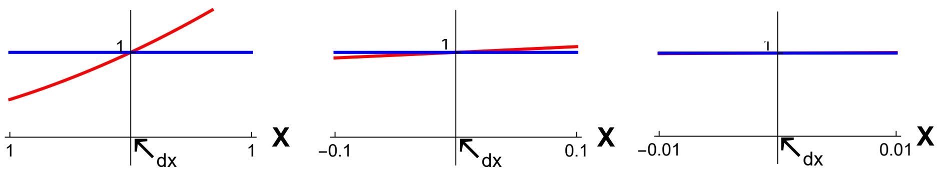


Finite Hyperreal Case $9 + 6\,dx + dx^2 \approx 9$ because

$$\frac{9+6\,dx+dx^2}{9} = 1 + \tfrac{2}{3}dx + \tfrac{dx^2}{9} = 1 + \epsilon.$$

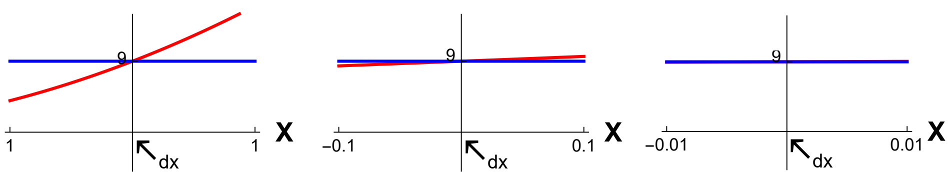
Graphically, compare ratio with 1:

$$\frac{9+6\,dx+dx^2}{9},\ 1$$



Graphically, compare individually:

$$9 + 6\,dx + dx^2,\ 9$$

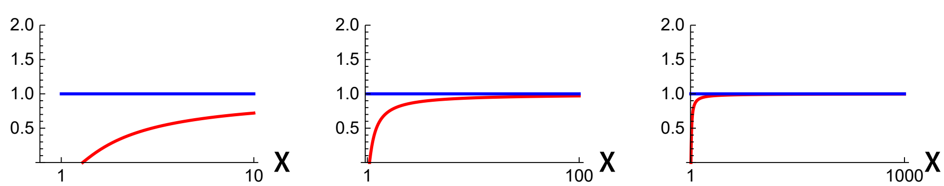


Infinite Case $X^2 - 3X + 2 \approx X^2$ because

$$\frac{X^2-3X+2}{X^2} = 1 - \frac{3}{X} + \frac{2}{X^2} = 1 + \epsilon \text{ for } X \text{ a positive infinite number.}$$

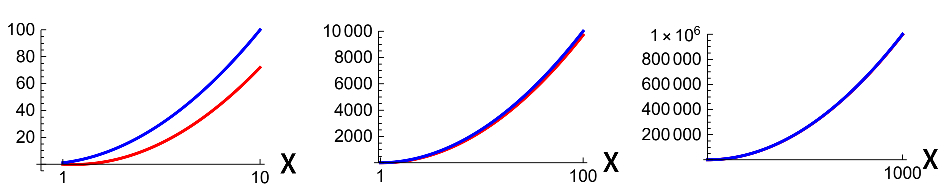
Graphically, compare ratio with 1:

$$\frac{X^2-3X+2}{X^2},\ 1$$



Graphically, compare individually:

$$X^2 - 3X + 2,\ X^2$$



4.2 Related Rates

When two quantities are related by an equation, knowing the value of one quantity can determine the value of the other. For instance, the circumference and radius of a circle are related by $C = 2\pi r$; knowing that $C = 6\pi$ in determines the radius must be 3 in.

The topic of **related rates** takes this one step further: knowing the *rate* at which one quantity is changing can determine the *rate* at which another changes.

We demonstrate the concepts of related rates through examples.

Example 4.2.1 Understanding related rates

The radius of a circle is growing at a rate of 5 in/hr. At what rate is the circumference growing?

Solution The circumference and radius of a circle are related by $C = 2\pi r$. We are given information about how the length of r changes with respect to time; that is, we are told $\frac{dr}{dt} = 5$in/hr. We want to know how the length of C changes with respect to time, i.e., we want to know $\frac{dC}{dt}$.

Implicitly differentiate both sides of $C = 2\pi r$ with respect to t:

$$\begin{aligned} C &= 2\pi r \\ \frac{d}{dt}\big(C\big) &= \frac{d}{dt}\big(2\pi r\big) \\ \frac{dC}{dt} &= 2\pi\frac{dr}{dt}. \end{aligned}$$

As we know $\frac{dr}{dt} = 5$in/hr, we know

$$\frac{dC}{dt} = 2\pi 5 = 10\pi \doteq 31.4\text{in/hr}.$$

Consider another, similar example.

Example 4.2.2 Finding related rates

Water streams out of a faucet at a rate of $2\text{in}^3/\text{s}$ onto a flat surface at a constant rate, forming a circular puddle that is $1/8$in deep.

1. At what rate is the area of the puddle growing?
2. At what rate is the radius of the circle growing?

Solution

1. We can answer this question two ways: using "common sense" or related rates. The common sense method states that the volume of the puddle is growing by 2in^3/s, where

   $$\text{volume of puddle} = \text{area of circle} \times \text{depth}.$$

   Since the depth is constant at $1/8$in, the area must be growing by 16in^2/s.

   This approach reveals the underlying related–rates principle. Let V and A represent the Volume and Area of the puddle. We know $V = A \times \frac{1}{8}$. Take the derivative of both sides with respect to t, employing implicit differentiation.

   $$\begin{aligned} V &= \frac{1}{8}A \\ \frac{d}{dt}\big(V\big) &= \frac{d}{dt}\left(\frac{1}{8}A\right) \\ \frac{dV}{dt} &= \frac{1}{8}\frac{dA}{dt} \end{aligned}$$

   As $\frac{dV}{dt} = 2$, we know $2 = \frac{1}{8}\frac{dA}{dt}$, and hence $\frac{dA}{dt} = 16$. Thus the area is growing by 16in^2/s.

2. To start, we need an equation that relates what we know to the radius. We just learned something about the surface area of the circular puddle, and we know $A = \pi r^2$. We should be able to learn about the rate at which the radius is growing with this information.

   Implicitly derive both sides of $A = \pi r^2$ with respect to t:

   $$\begin{aligned} A &= \pi r^2 \\ \frac{d}{dt}\big(A\big) &= \frac{d}{dt}\big(\pi r^2\big) \\ \frac{dA}{dt} &= 2\pi r\frac{dr}{dt} \end{aligned}$$

   Our work above told us that $\frac{dA}{dt} = 16$in^2/s. Solving for $\frac{dr}{dt}$, we have

   $$\frac{dr}{dt} = \frac{8}{\pi r}.$$

   Note how our answer is not a number, but rather a function of r. In other words, *the rate at which the radius is growing depends on how big the* *circle already is.* If the circle is very large, adding 2in^3 of water will not make the circle much bigger at all. If the circle is dime–sized, adding the same amount of water will make a radical change in the radius of the circle.

In some ways, our problem was (intentionally) ill–posed. We need to specify a current radius in order to know a rate of change. When the puddle has a radius of 10in, the radius is growing at a rate of

$$\frac{dr}{dt} = \frac{8}{10\pi} = \frac{4}{5\pi} \doteq 0.25\text{in/s}.$$

Example 4.2.3 Studying related rates

Radar guns measure the rate of distance change between the gun and the object it is measuring. For instance, a reading of "55mph" means the object is moving away from the gun at a rate of 55 miles per hour, whereas a measurement of "-25mph" would mean that the object is approaching the gun at a rate of 25 miles per hour.

If the radar gun is moving (say, attached to a police car) then radar readouts are only immediately understandable if the gun and the object are moving along the same line. If a police officer is traveling 60mph and gets a readout of 15mph, he knows that the car ahead of him is moving away at a rate of 15 miles an hour, meaning the car is traveling 75mph. (This straight–line principle is one reason officers park on the side of the highway and try to shoot straight back down the road. It gives the most accurate reading.)

Suppose an officer is driving due north at 30 mph and sees a car moving due east, as shown in Figure 4.2.1. Using his radar gun, he measures a reading of 20mph. By using landmarks, he believes both he and the other car are about 1/2 mile from the intersection of their two roads.

If the speed limit on the other road is 55mph, is the other driver speeding?

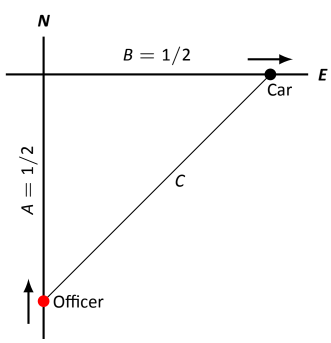


Figure 4.2.1: A sketch of a police car (at bottom) attempting to measure the speed of a car (at right) in Example 4.2.3.

Solution Using the diagram in Figure 4.2.1, let's label what we know about the situation. As both the police officer and other driver are 1/2 mile from the intersection, we have $A = 1/2$, $B = 1/2$, and through the Pythagorean Theorem, $C = 1/\sqrt{2} \doteq 0.707$.

We know the police officer is traveling at 30mph; that is, $\frac{dA}{dt} = -30$. The reason this rate of change is negative is that A is getting smaller; the distance between the officer and the intersection is shrinking. The radar measurement is $\frac{dC}{dt} = 20$. We want to find $\frac{dB}{dt}$.

We need an equation that relates B to A and/or C. The Pythagorean Theorem is a good choice: $A^2 + B^2 = C^2$. Differentiate both sides with respect to t:

$$A^2 + B^2 = C^2$$
$$\frac{d}{dt}\left(A^2 + B^2\right) = \frac{d}{dt}\left(C^2\right)$$
$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2C\frac{dC}{dt}$$

We have values for everything except $\frac{dB}{dt}$. Solving for this we have

$$\frac{dB}{dt} = \frac{C\frac{dC}{dt} - A\frac{dA}{dt}}{B} \doteq 58.28\text{mph}.$$

The other driver appears to be speeding slightly.

Note: Example 4.2.3 is both interesting and impractical. It highlights the difficulty in using radar in a non–linear fashion, and explains why "in real life" the police officer would follow the other driver to determine their speed, and not pull out pencil and paper.
The principles here are important, though. Many automated vehicles make judgments about other moving objects based on perceived distances, radar–like measurements and the concepts of related rates.

Example 4.2.4 Studying related rates
A camera is placed on a tripod 10ft from the side of a road. The camera is to turn to track a car that is to drive by at 100mph for a promotional video. The video's planners want to know what kind of motor the tripod should be equipped with in order to properly track the car as it passes by. Figure 4.2.2 shows the proposed setup.

How fast must the camera be able to turn to track the car?

Solution We seek information about how fast the camera is to *turn*; therefore, we need an equation that will relate an angle θ to the position of the camera and the speed and position of the car.

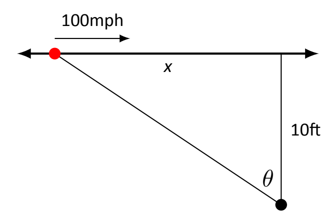


Figure 4.2.2: Tracking a speeding car (at left) with a rotating camera.

Figure 4.2.2 suggests we use a trigonometric equation. Letting x represent the distance the car is from the point on the road directly in front of the camera, we have

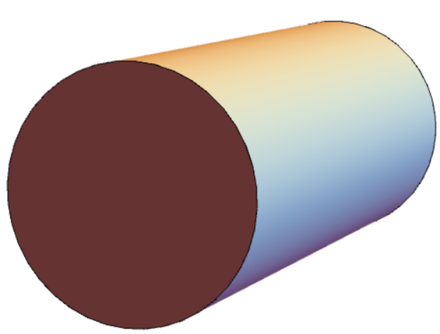
$$\tan\theta = \frac{x}{10}. \tag{4.1}$$

As the car is moving at 100mph, we have $\frac{dx}{dt} = -100$mph (as in the last example, since x is getting smaller as the car travels, $\frac{dx}{dt}$ is negative). We need to convert the measurements so they use the same units; rewrite -100mph in terms of ft/s:

$$\frac{dx}{dt} = -100\frac{\text{m}}{\text{hr}} = -100\frac{\text{m}}{\text{hr}}\cdot 5280\frac{\text{ft}}{\text{m}}\cdot\frac{1}{3600}\frac{\text{hr}}{\text{s}} = -146.\overline{6}\text{ft/s}.$$

Now take the derivative of both sides of Equation (4.1) using implicit differentiation:

Example Infinitesimal Analysis



A water tank is a horizontal cylinder of radius 5 feet and length 15 feet. Water is poured into the tank at the rate $8\ \frac{\text{feet}^3}{\text{minute}}$. How fast is the water level rising when the water is 9 feet deep?
Unfortunately you do not know the formula for the volume as a function of the water depth, V = V(y). You will learn it in the next calculus course: it is quite complicated. Fortunately, you can easily work the problem using infinitesimal analysis! See the solution on the next page.

$$\tan\theta = \frac{x}{10}$$
$$\frac{d}{dt}\big(\tan\theta\big) = \frac{d}{dt}\left(\frac{x}{10}\right)$$
$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$$
$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{10}\frac{dx}{dt} \qquad (4.2)$$

We want to know the fastest the camera has to turn. Common sense tells us this is when the car is directly in front of the camera (i.e., when $\theta = 0$). Our mathematics bears this out. In Equation (4.2) we see this is when $\cos^2\theta$ is largest; this is when $\cos\theta = 1$, or when $\theta = 0$.

With $\frac{dx}{dt} \doteq -146.67\text{ft/s}$, we have

$$\frac{d\theta}{dt} = -\frac{1\text{rad}}{10\text{ft}}146.67\text{ft/s} = -14.667\text{radians/s}.$$

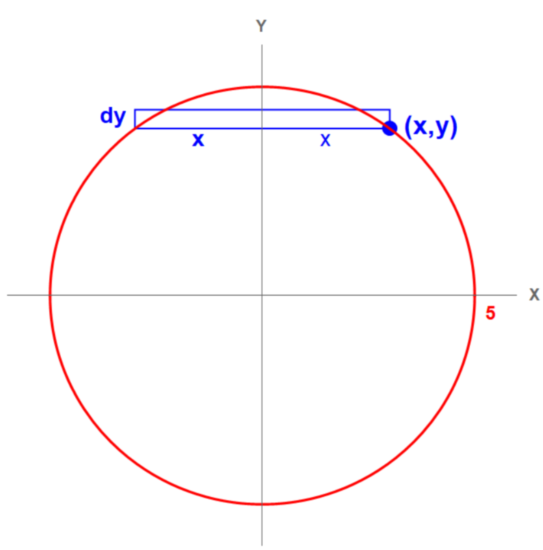
We find that $\frac{d\theta}{dt}$ is negative; this matches our diagram in Figure 4.2.2 for θ is getting smaller as the car approaches the camera.

What is the practical meaning of -14.667radians/s? Recall that 1 circular revolution goes through 2π radians, thus 14.667rad/s means $14.667/(2\pi) \doteq$ 2.33 revolutions per second. The negative sign indicates the camera is rotating in a clockwise fashion.

We introduced the derivative as a function that gives the slopes of tangent lines of functions. This chapter emphasizes using the derivative in other ways. Newton's Method uses the derivative to approximate roots of functions; this section stresses the "rate of change" aspect of the derivative to find a relationship between the rates of change of two related quantities.

In the next section we use Extreme Value concepts to *optimize* quantities.

Solution by infinitesimal analysis



Solution From the graph, the volume of the infinitesimal layer is

$$dV \approx 2x\,(16)\,dy \approx 32\sqrt{25-y^2}\,dy$$
$$dy \approx \frac{dV}{32\sqrt{25-y^2}}$$
$$\frac{dy}{dt} \approx \left.\frac{\frac{dV}{dt}}{32\sqrt{25-y^2}}\right|_{y=4,\ \frac{dV}{dt}=8} = \frac{1}{12}\,\frac{\text{meter}}{\text{minute}}.$$

Exercises 4.2

Terms and Concepts

1. T/F: Implicit differentiation is often used when solving "related rates" type problems.

2. T/F: A study of related rates is part of the standard police officer training.

Problems

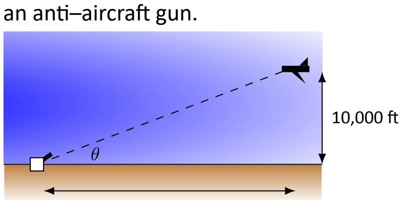
3. Water flows onto a flat surface at a rate of $5\text{cm}^3/\text{s}$ forming a circular puddle 10mm deep. How fast is the radius growing when the radius is:
   (a) 1 cm?
   (b) 10 cm?
   (c) 100 cm?

4. A circular balloon is inflated with air flowing at a rate of $10\text{cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the radius is:
   (a) 1 cm?
   (b) 10 cm?
   (c) 100 cm?

5. Consider the traffic situation introduced in Example 4.2.3. How fast is the "other car" traveling if the officer and the other car are each 1/2 mile from the intersection, the other car is traveling *due west*, the officer is traveling north at 50mph, and the radar reading is -80mph?

6. Consider the traffic situation introduced in Example 4.2.3. Calculate how fast the "other car" is traveling in each of the following situations.
   (a) The officer is traveling due north at 50mph and is 1/2 mile from the intersection, while the other car is 1 mile from the intersection traveling west and the radar reading is -80mph?
   (b) The officer is traveling due north at 50mph and is 1 mile from the intersection, while the other car is 1/2 mile from the intersection traveling west and the radar reading is -80mph?

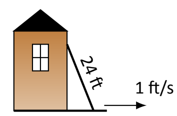
7. An F-22 aircraft is flying at 500mph with an elevation of 10,000ft on a straight–line path that will take it directly over an anti–aircraft gun.



   How fast must the gun be able to turn to accurately track the aircraft when the plane is:
   (a) 1 mile away?
   (b) 1/5 mile away?
   (c) Directly overhead?

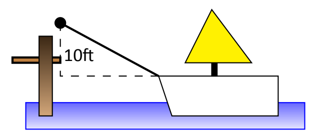
8. An F-22 aircraft is flying at 500mph with an elevation of 100ft on a straight–line path that will take it directly over an anti–aircraft gun as in Exercise 7 (note the lower elevation here).
   How fast must the gun be able to turn to accurately track the aircraft when the plane is:
   (a) 1000 feet away?
   (b) 100 feet away?
   (c) Directly overhead?

9. A 24ft. ladder is leaning against a house while the base is pulled away at a constant rate of 1ft/s.



   At what rate is the top of the ladder sliding down the side of the house when the base is:
   (a) 1 foot from the house?
   (b) 10 feet from the house?
   (c) 23 feet from the house?
   (d) 24 feet from the house?

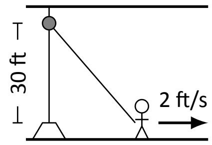
10. A boat is being pulled into a dock at a constant rate of 30ft/min by a winch located 10ft above the deck of the boat.



   At what rate is the boat approaching the dock when the boat is:
   (a) 50 feet out?
   (b) 15 feet out?
   (c) 1 foot from the dock?
   (d) What happens when the length of rope pulling in the boat is less than 10 feet long?

11. An inverted cylindrical cone, 20ft deep and 10ft across at the top, is being filled with water at a rate of $10\text{ft}^3/\text{min}$. At what rate is the water rising in the tank when the depth of the water is:
   (a) 1 foot?
   (b) 10 feet?
   (c) 19 feet?
   How long will the tank take to fill when starting at empty?

12. A rope, attached to a weight, goes up through a pulley at the ceiling and back down to a worker. The man holds the rope at the same height as the connection point between rope and weight.



Suppose the man stands directly next to the weight (i.e., a total rope length of 60 ft) and begins to walk away at a rate of 2ft/s. How fast is the weight rising when the man has walked:

(a) 10 feet?

(b) 40 feet?

How far must the man walk to raise the weight all the way to the pulley?

13. Consider the situation described in Exercise 12. Suppose the man starts 40ft from the weight and begins to walk away at a rate of 2ft/s.

(a) How long is the rope?

(b) How fast is the weight rising after the man has walked 10 feet?

(c) How fast is the weight rising after the man has walked 30 feet?

(d) How far must the man walk to raise the weight all the way to the pulley?

14. A hot air balloon lifts off from ground rising vertically. From 100 feet away, a 5' woman tracks the path of the balloon. When her sightline with the balloon makes a 45° angle with the horizontal, she notes the angle is increasing at about 5°/min.

(a) What is the elevation of the balloon?

(b) How fast is it rising?

15. A company that produces landscaping materials is dumping sand into a conical pile. The sand is being poured at a rate of 5ft^3/sec; the physical properties of the sand, in conjunction with gravity, ensure that the cone's height is roughly 2/3 the length of the diameter of the circular base.

How fast is the cone rising when it has a height of 30 feet?

16. In the cylindrical water tank problem, how fast is the water level rising when the tank is half full?

17. A spherical petroleum tank has a radius of 8 feet. Liquid is removed out of the bottom at the rate of $1\ \frac{\text{foot}^3}{\text{minute}}$.

a. How fast is the level decreasing when the tank is half full?

b. How fast is the level decreasing when the level is at 4 feet?

Use infinitesimal methods.

18. A medicinal CBD oil tank has the form of an inverted truncated pyramid 40 centimeters high, the top being a square with side 20 centimeters and the bottom a square of side 10 centimeters. Oil is withdrawn at the rate of $10\ \frac{\text{cm}^3}{\text{second}}$.

a. How fast is the level decreasing when the tank is full?

b. How fast is the level decreasing when the oil level is at 20 centimeters?

Use infinitesimal methods.

19. A parabolic tank is obtained by rotating the parabola $y = x^2$, $0 \le x \le 10$ cm, about the y-axis. Water is being poured into the tank at the rate 100 cm^3/min. How fast is the water level rising when the level is 10cm?

Solutions 4.2

1. T
2. F
3. (a) $5/(2\pi) \doteq 0.796$cm/s
   (b) $1/(4\pi) \doteq 0.0796$ cm/s
   (c) $1/(40\pi) \doteq 0.00796$ cm/s
4. (a) $5/(2\pi) \doteq 0.796$cm/s
   (b) $1/(40\pi) \doteq 0.00796$ cm/s
   (c) $1/(4000\pi) \doteq 0.0000796$ cm/s
5. 63.14mph
6. (a) 64.44 mph
   (b) 78.89 mph
7. Due to the height of the plane, the gun does not have to rotate very fast.
   (a) 0.0573 rad/s
   (b) 0.0725 rad/s
   (c) In the limit, rate goes to 0.0733 rad/s
8. Due to the height of the plane, the gun does not have to rotate very fast.
   (a) 0.073 rad/s
   (b) 3.66 rad/s (about 1/2 revolution/sec)
   (c) In the limit, rate goes to 7.33 rad/s (more than 1 revolution/sec)
9. (a) 0.04 ft/s
   (b) 0.458 ft/s
   (c) 3.35 ft/s
   (d) Not defined; as the distance approaches 24, the rates approaches ∞.
10. (a) 30.59 ft/min
    (b) 36.1 ft/min
    (c) 301 ft/min
    (d) The boat no longer floats as usual, but is being pulled up by the winch (assuming it has the power to do so).
11. (a) 50.92 ft/min
    (b) 0.509 ft/min
    (c) 0.141 ft/min

    As the tank holds about 523.6ft^3, it will take about 52.36 minutes.
12. (a) 0.63 ft/sec
    (b) 1.6 ft/sec

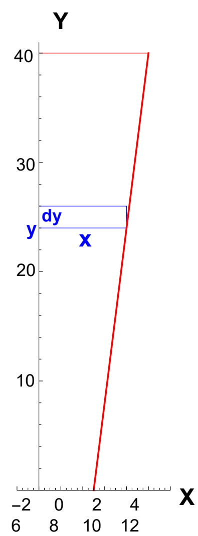
    About 52 ft.
13. (a) The rope is 80ft long.
    (b) 1.71 ft/sec
    (c) 1.84 ft/sec
    (d) About 34 feet.
14. (a) The balloon is 105ft in the air.
    (b) The balloon is rising at a rate of 17.45ft/min. (Hint: convert all angles to radians.)
15. The cone is rising at a rate of 0.003ft/s.

16. $\frac{dy}{dt} = \frac{1}{20} \frac{\text{meter}}{\text{minute}}$

17 a. $\frac{dy}{dt} = -\frac{1}{25\pi} \frac{\text{feet}}{\text{minute}}$

b. $\frac{dy}{dt} = -\frac{1}{9\pi} \frac{\text{feet}}{\text{minute}}$

18.



Front-right quarter of tank shown.

$y = 8x - 40$

$dV = (2x)^2\, dy$

$= 4x^2 dy$

$= 4\left(\frac{y}{8} + 5\right)^2 dy$

$\frac{dV}{dt} = 4\left(\frac{y}{8} + 5\right)^2 \frac{dy}{dt}$

$\frac{dy}{dt} = \frac{dy/dt}{4\left(\frac{y}{8}+5\right)^2}$

$= \frac{-10}{4\,(5+5)^2}$

$= -\frac{1}{40} \frac{\text{cm}}{\text{second}}$

19. $\frac{dy}{dt} = \frac{10}{\pi} \frac{\text{cm}}{\text{minute}}$

4.3 Optimization

In Section 3.1 we learned about extreme values – the largest and smallest values a function attains on an interval. We motivated our interest in such values by discussing how it made sense to want to know the highest/lowest values of a stock, or the fastest/slowest an object was moving. In this section we apply the concepts of extreme values to solve "word problems," i.e., problems stated in terms of situations that require us to create the appropriate mathematical framework in which to solve the problem.

We start with a classic example which is followed by a discussion of the topic of optimization.

Example 4.3.1 Optimization: perimeter and area

A man has 100 feet of fencing, a large yard, and a small dog. He wants to create a rectangular enclosure for his dog with the fencing that provides the maximal area. What dimensions provide the maximal area?

Solution One can likely guess the correct answer – that is great. We will proceed to show how calculus can provide this answer in a context that proves this answer is correct.

It helps to make a sketch of the situation. Our enclosure is sketched twice in Figure 4.3.1, either with green grass and nice fence boards or as a simple rectangle. Either way, drawing a rectangle forces us to realize that we need to know the dimensions of this rectangle so we can create an area function – after all, we are trying to maximize the area.

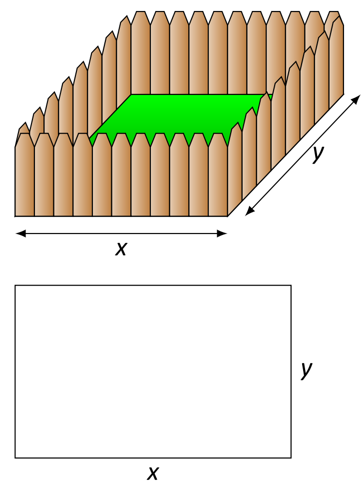


Figure 4.3.1: A sketch of the enclosure in Example 4.3.1.

We let x and y denote the lengths of the sides of the rectangle. Clearly,

$$\text{Area} = xy.$$

We do not yet know how to handle functions with 2 variables; we need to reduce this down to a single variable. We know more about the situation: the man has 100 feet of fencing. By knowing the perimeter of the rectangle must be 100, we can create another equation:

$$\text{Perimeter} = 100 = 2x + 2y.$$

We now have 2 equations and 2 unknowns. In the latter equation, we solve for y:

$$y = 50 - x.$$

Now substitute this expression for y in the area equation:

$$\text{Area} = A(x) = x(50 - x).$$

Note we now have an equation of one variable; we can truly call the Area a function of x.

This function only makes sense when $0 \leq x \leq 50$, otherwise we get negative values of area. So we find the extreme values of $A(x)$ on the interval $[0, 50]$.

To find the critical points, we take the derivative of $A(x)$ and set it equal to 0, then solve for x.

$$\begin{aligned} A(x) &= x(50 - x) \\ &= 50x - x^2 \\ A'(x) &= 50 - 2x \end{aligned}$$

We solve $50 - 2x = 0$ to find $x = 25$; this is the only critical point. We evaluate $A(x)$ at the endpoints of our interval and at this critical point to find the extreme values; in this case, all we care about is the maximum.

Clearly $A(0) = 0$ and $A(50) = 0$, whereas $A(25) = 625\text{ft}^2$. This is the maximum. Since we earlier found $y = 50 - x$, we find that y is also 25. Thus the dimensions of the rectangular enclosure with perimeter of 100 ft. with maximum area is a square, with sides of length 25 ft.

This example is very simplistic and a bit contrived. (After all, most people create a design then buy fencing to meet their needs, and not buy fencing and plan later.) But it models well the necessary process: create equations that describe a situation, reduce an equation to a single variable, then find the needed extreme value.

"In real life," problems are much more complex. The equations are often *not* reducible to a single variable (hence multi–variable calculus is needed) and the equations themselves may be difficult to form. Understanding the principles here will provide a good foundation for the mathematics you will likely encounter later.

We outline here the basic process of solving these optimization problems.

Key Idea 4.3.1 Solving Optimization Problems

1. Understand the problem. Clearly identify what quantity is to be maximized or minimized. Make a sketch if helpful.

2. Create equations relevant to the context of the problem, using the information given. (One of these should describe the quantity to be optimized. We'll call this the *fundamental equation.*)

3. If the fundamental equation defines the quantity to be optimized as a function of more than one variable, reduce it to a single variable function using substitutions derived from the other equations.

(continued). . .

Key Idea 4.3.1 Solving Optimization Problems – Continued

4. Identify the domain of this function, keeping in mind the context of the problem.
5. Find the extreme values of this function on the determined domain.
6. Identify the values of all relevant quantities of the problem.

We will use Key Idea 4.3.1 in a variety of examples.

Example 4.3.2 Optimization: perimeter and area

Here is another classic calculus problem: A woman has a 100 feet of fencing, a small dog, and a large yard that contains a stream (that is mostly straight). She wants to create a rectangular enclosure with maximal area that uses the stream as one side. (Apparently her dog won't swim away.) What dimensions provide the maximal area?

SOLUTION We will follow the steps outlined by Key Idea 4.3.1.

1. We are maximizing *area*. A sketch of the region will help; Figure 4.3.2 gives two sketches of the proposed enclosed area. A key feature of the sketches is to acknowledge that one side is not fenced.
2. We want to maximize the area; as in the example before,

   $$\text{Area} = xy.$$

   This is our fundamental equation. This defines area as a function of two variables, so we need another equation to reduce it to one variable.

   We again appeal to the perimeter; here the perimeter is

   $$\text{Perimeter} = 100 = x + 2y.$$

   Note how this is different than in our previous example.
3. We now reduce the fundamental equation to a single variable. In the perimeter equation, solve for y: $y = 50 - x/2$. We can now write Area as

   $$\text{Area} = A(x) = x(50 - x/2) = 50x - \frac{1}{2}x^2.$$

   Area is now defined as a function of one variable.

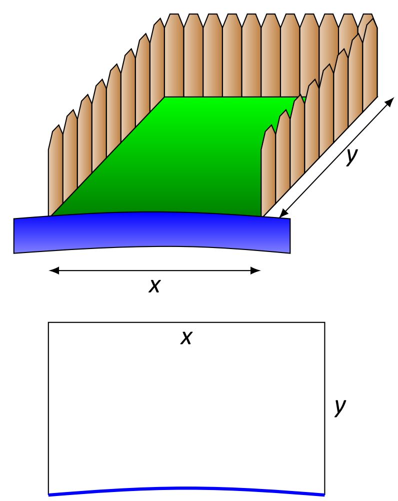


Figure 4.3.2: A sketch of the enclosure in Example 4.3.2.

4. We want the area to be nonnegative. Since $A(x) = x(50 - x/2)$, we want $x \geq 0$ and $50 - x/2 \geq 0$. The latter inequality implies that $x \leq 100$, so $0 \leq x \leq 100$.
5. We now find the extreme values. At the endpoints, the minimum is found, giving an area of 0.

   Find the critical points. We have $A'(x) = 50 - x$; setting this equal to 0 and solving for x returns $x = 50$. This gives an area of

   $$A(50) = 50(25) = 1250.$$

6. We earlier set $y = 50 - x/2$; thus $y = 25$. Thus our rectangle will have two sides of length 25 and one side of length 50, with a total area of 1250 ft^2.

Keep in mind as we do these problems that we are practicing a *process*; that is, we are learning to turn a situation into a system of equations. These equations allow us to write a certain quantity as a function of one variable, which we then optimize.

Example 4.3.3 Optimization: minimizing cost

A power line needs to be run from a power station located on the beach to an offshore facility. Figure 4.3.3 shows the distances between the power station to the facility.

It costs \$50/ft. to run a power line along the land, and \$130/ft. to run a power line under water. How much of the power line should be run along the land to minimize the overall cost? What is the minimal cost?

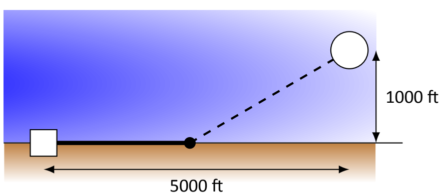


Figure 4.3.3: Running a power line from the power station to an offshore facility with minimal cost in Example 4.3.3.

SOLUTION We will follow the strategy of Key Idea 4.3.1 implicitly, without specifically numbering steps.

There are two immediate solutions that we could consider, each of which we will reject through "common sense." First, we could minimize the distance by directly connecting the two locations with a straight line. However, this requires that all the wire be laid underwater, the most costly option. Second, we could minimize the underwater length by running a wire all 5000 ft. along the beach, directly across from the offshore facility. This has the undesired effect of having the longest distance of all, probably ensuring a non–minimal cost.

The optimal solution likely has the line being run along the ground for a while, then underwater, as the figure implies. We need to label our unknown distances – the distance run along the ground and the distance run underwater. Recognizing that the underwater distance can be measured as the hypotenuse of a right triangle, we choose to label the distances as shown in Figure 4.3.4.

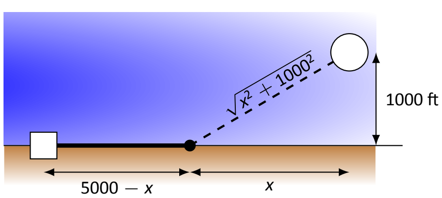


Figure 4.3.4: Labeling unknown distances in Example 4.3.3.

By choosing x as we did, we make the expression under the square root simple. We now create the cost function.

$$\begin{array}{rcccc} \text{Cost} & = & \text{land cost} & + & \text{water cost} \\ & & \$50 \times \text{land distance} & + & \$130 \times \text{water distance} \\ & & 50(5000-x) & + & 130\sqrt{x^2+1000^2}. \end{array}$$

So we have $c(x) = 50(5000-x) + 130\sqrt{x^2+1000^2}$. This function only makes sense on the interval $[0, 5000]$. While we are fairly certain the endpoints will not give a minimal cost, we still evaluate $c(x)$ at each to verify.

$$c(0) = 380,000 \qquad c(5000) \doteq 662,873.$$

We now find the critical values of $c(x)$. We compute $c'(x)$ as

$$c'(x) = -50 + \frac{130x}{\sqrt{x^2+1000^2}}.$$

Recognize that this is never undefined. Setting $c'(x) = 0$ and solving for x, we have:

$$\begin{aligned} -50 + \frac{130x}{\sqrt{x^2+1000^2}} &= 0 \\ \frac{130x}{\sqrt{x^2+1000^2}} &= 50 \\ \frac{130^2x^2}{x^2+1000^2} &= 50^2 \\ 130^2x^2 &= 50^2(x^2+1000^2) \\ 130^2x^2 - 50^2x^2 &= 50^2 \cdot 1000^2 \\ (130^2-50^2)x^2 &= 50,000^2 \\ x^2 &= \frac{50,000^2}{130^2-50^2} \\ x &= \frac{50,000}{\sqrt{130^2-50^2}} \\ x &= \frac{50,000}{120} = \frac{1250}{3} \doteq 416.67. \end{aligned}$$

Evaluating $c(x)$ at $x = 416.67$ gives a cost of about \$370,000. The distance the power line is laid along land is $5000 - 416.67 = 4583.33$ ft., and the underwater distance is $\sqrt{416.67^2+1000^2} \doteq 1083$ ft.

In the exercises you will see a variety of situations that require you to combine problem–solving skills with calculus. Focus on the *process*; learn how to form equations from situations that can be manipulated into what you need. Eschew memorizing how to do "this kind of problem" as opposed to "that kind of problem." Learning a process will benefit one far longer than memorizing a specific technique.

The next section introduces our final application of the derivative: *differentials*. Given $y = f(x)$, they offer a method of approximating the change in y after x changes by a small amount.

1. T/F: An "optimization problem" is essentially an "extreme values" problem in a "story problem" setting.

2. T/F: This section teaches one to find the extreme values of a function that has more than one variable.

Problems

3. Find the maximum product of two numbers (not necessarily integers) that have a sum of 100.

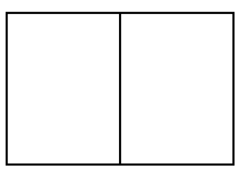
4. Find the minimum sum of two positive numbers whose product is 500.

5. Find the maximum sum of two positive numbers whose product is 500.

6. Find the maximum sum of two numbers, each of which is in $[0, 300]$ whose product is 500.

7. Find the maximal area of a right triangle with hypotenuse of length 1.

8. A rancher has 1000 feet of fencing in which to construct adjacent, equally sized rectangular pens. What dimensions should these pens have to maximize the enclosed area?



9. A standard soda can is roughly cylindrical and holds 355cm^3 of liquid. What dimensions should the cylinder be to minimize the material needed to produce the can? Based on your dimensions, determine whether or not the standard can is produced to minimize the material costs.

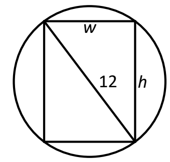
10. Find the dimensions of a cylindrical can with a volume of 206in^3 that minimizes the surface area.

    The "#10 can"is a standard sized can used by the restaurant industry that holds about 206in^3 with a diameter of 6 2/16in and height of 7in. Does it seem these dimensions were chosen with minimization in mind?

11. The United States Postal Service charges more for boxes whose combined length and girth exceeds 108" (the "length" of a package is the length of its longest side; the girth is the perimeter of the cross section, i.e., $2w + 2h$).

    What is the maximum volume of a package with a square cross section ($w = h$) that does not exceed the 108" standard?

12. The strength S of a wooden beam is directly proportional to its cross sectional width w and the square of its height h; that is, $S = kwh^2$ for some constant k.



    Given a circular log with diameter of 12 inches, what sized beam can be cut from the log with maximum strength?

13. A power line is to be run to an offshore facility in the manner described in Example 4.3.3. The offshore facility is 2 miles at sea and 5 miles along the shoreline from the power plant. It costs \$50,000 per mile to lay a power line underground and \$80,000 to run the line underwater.

    How much of the power line should be run underground to minimize the overall costs?

14. A power line is to be run to an offshore facility in the manner described in Example 4.3.3. The offshore facility is 5 miles at sea and 2 miles along the shoreline from the power plant. It costs \$50,000 per mile to lay a power line underground and \$80,000 to run the line underwater.

    How much of the power line should be run underground to minimize the overall costs?

15. A woman throws a stick into a lake for her dog to fetch; the stick is 20 feet down the shore line and 15 feet into the water from there. The dog may jump directly into the water and swim, or run along the shore line to get closer to the stick before swimming. The dog runs about 22ft/s and swims about 1.5ft/s.

    How far along the shore should the dog run to minimize the time it takes to get to the stick? (Hint: the figure from Example 4.3.3 can be useful.)

16. A woman throws a stick into a lake for her dog to fetch; the stick is 15 feet down the shore line and 30 feet into the water from there. The dog may jump directly into the water and swim, or run along the shore line to get closer to the stick before swimming. The dog runs about 22ft/s and swims about 1.5ft/s.

    How far along the shore should the dog run to minimize the time it takes to get to the stick? *(Google "calculus dog" to learn more about a dog's ability to minimize times.)*

17. What are the dimensions of the rectangle with largest area that can be drawn inside the unit circle?

Solutions 4.3

1. T
2. F
3. 2500; the two numbers are each 50.
4. The minimum sum is $2\sqrt{500}$; the two numbers are each $\sqrt{500}$.
5. There is no maximum sum; the fundamental equation has only 1 critical value that corresponds to a minimum.
6. The only critical point of the fundamental equation corresponds to a minimum; to find maximum, we check the endpoints.

   If one number is 300, the other number y satisfies $300y = 500$; $y = 5/3$. Thus the sum is $300 + 5/3$.

   The other endpoint, 0, is not feasible as we cannot solve $0 \cdot y = 500$ for y. In fact, if $0 < x < 5/3$, then $x \cdot y = 500$ forces $y > 300$, which is not a feasible solution.

   Hence the maximum sum is $301.\overline{6}$.
7. Area = 1/4, with sides of length $1/\sqrt{2}$.
8. Each pen should be $500/3 \doteq 166.67$ feet by 125 feet.
9. The radius should be about 3.84cm and the height should be $2r = 7.67$cm. No, this is not the size of the standard can.
10. The radius should be about 3.2in and the height should be $2r = 6.4$in. As the #10 is not a perfect cylinder (with extra material to aid in stacking, etc.), the dimensions are close enough to assume that minimizing surface area was a consideration.
11. The height and width should be 18 and the length should be 36, giving a volume of $11,664\text{in}^3$.
12. $w = 4\sqrt{3}$, $h = 4\sqrt{6}$
13. $5 - 10/\sqrt{39} \doteq 3.4$ miles should be run underground, giving a minimum cost of $374,899.96.
14. The power line should be run directly to the off shore facility, skipping any underground, giving a cost of about $430,813.
15. The dog should run about 19 feet along the shore before starting to swim.
16. The dog should run about 13 feet along the shore before starting to swim.
17. The largest area is 2 formed by a square with sides of length $\sqrt{2}$.

4.4 Differentials - Preliminary Overview

The derivative and the differential provide simple, effective ways of approximating a function.

Tangent Line Approximations

The tangent line to $y = f(x)$ at $x = a$ is

$$y = f(a) + f'(a)(x - a).$$

The tangent line approximation to $y = f(x)$ at $x = a$ is

$$\boxed{f(x) \doteq f(a) + f'(a)(x - a)}.$$

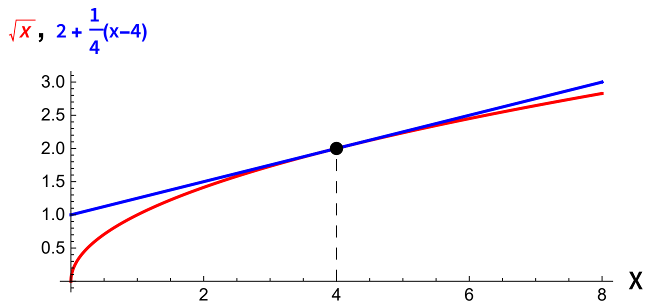
Example

a. Find the tangent line approximation to $y = \sqrt{x}$ at $x = 4$.

$$f(x) = \sqrt{x},\ f(4) = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x}},\quad f'(x) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\Longrightarrow \sqrt{x} \doteq 2 + \tfrac{1}{4}(x - 4) \quad \text{near } x = 4.$$



b. Use the above approximation to to calculate $\sqrt{4.08}$.

$$\sqrt{4.08} \doteq 2 + \tfrac{1}{4}(4.08 - 4) = 2.02$$

Exact answer

$$\sqrt{4.08} = 2.0199009876724 \cdots$$

Differentials

Definition The ***differential*** of the function $y = f(x)$:

1. dx is an infinitesimal
2. $dy = f'(x)dx$.

Historically, the differential played a central, beginning role in calculus. You find the differential of a quantity y, often a simple equation, by examining its behavior over a short interval. If you divided the result by dx, you get the ***slope***. If you divided it by dt, you get the ***growth rate***. If you did a suitable sum of the dy's you would its total change (later called its ***definite integral***).

Now-a-days you usually see in textbooks that dx is taken to be any real number $-\infty < dx < +\infty$. This bit of silliness is to allow you, without knowing about infinitesimals, later on to feel comfortable with doing seemingly illegal things in standard calculus like making a change of variables in integrals, setting up integrals for applications or separating variables in differential equations.

You can readily change any derivative formula $\frac{dy}{dx} = f'(x)$ into its differential form $dy = f'(x)dx$ if you want. (The $=$ should really be a $\approx$, but we won't be compulsive about this.)

General Formulas
Let $u = u(x)$ and $v = v(x)$ be differentiable.

I. Constant Multiple Rule $d(c\,u) = c\,du$

II. Sum Rule $d(u + v) = du + dv$

III. Product Rule $d(u\,v) = v\,du + u\,dv$

IV. Quotient Rule $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

V. Chain Rule $d(u(v)) = u'(v)\,dv$

Special Differential Formulas

$d(c) = 0$

$d(\sin x) = \cos x\,dx$

$d(\tan x) = \sec^2 x\,dx$

$d(\sec x) = \sec x \tan x\,dx$

$d(e^x) = e^x\,dx$

$d(x^n) = n\,x^{n-1}\,dx$

$d(\cos x) = -\sin x\,dx$

$d(\cot x) = -\csc^2 x\,dx$

$d(\csc x) = -\csc x \cot x\,dx$

$d(\ln x) = \frac{dx}{x}$

Example $d(x^2 + 3\sin x) = (2x + 3\cos x)dx$

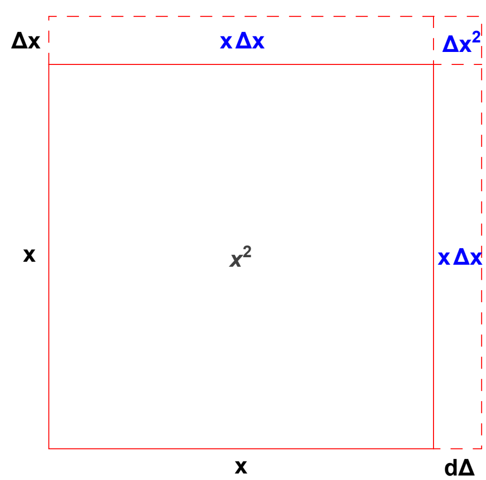
Differential Approximations

The differential approximation associated with $dy \approx f'(x)\,dx$ is

$$\boxed{\Delta y \doteq f'(x)\,\Delta x}$$

The differential approximation says that, near x, dy is approximately proportional to dx, everyones favorite relationship between two quantities.

Example Illustrate the difference between the exact ΔA and the approximate ΔA for a square.



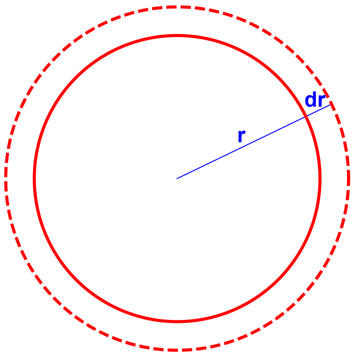
$A = x^2$

exact $\Longrightarrow \Delta A = (x + \Delta x)^2 - x^2 = (x^2 + 2x\,\Delta x + \Delta x^2) - x^2 = 2x\,\Delta x + \Delta x^2$

approximate $\Longrightarrow \Delta A \doteq 2x\,\Delta x$

$\Longrightarrow$ very close if Δx is a small real number!

Example How much paint is required to paint a sphere of radius 1 meter with a coating 1 mm thick. Use $V = \frac{4}{3}\pi r^3$. 1 meter = 1000 mm.



$\Delta V \doteq 4\pi r^2 \Delta r$

$= 4\pi 1000^2 (1)$

$= 4\pi$ litres

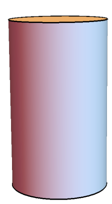
$\doteq$ 3 gallons of paint

Error Analysis

Suppose $y = f(x)$. If the error in measuring x is Δx, then the error in calculating y is $y \pm \Delta y = f(x \pm \Delta x)$ where

$$\boxed{\Delta y \doteq f'(x)\,\Delta x}.$$

Example The height of a can is $h = 30$ cm. Its radius is measured to be 10 ± 0.1 cm.



What is the volume and possible error in its calculated volume?

$V = \pi r^2 h = \pi 10^2 \cdot 30 = 3000\pi \text{cm}^3$ **Nominal Volume**

$\Longrightarrow \Delta V = 2\pi r h \Delta r = 2\pi 10 \cdot 20 \cdot 0.1 = 40\pi \text{cm}^3$ **Possible Error**

$\Longrightarrow V = (3000\pi \pm 40\pi)\,\text{cm}^3$ **Volume with Possible Error**

Note: in the APEX Calculus exercises, use the notations and definitions above.

4.4 Differentials - Further Examples

Historically differentials were invented to discover physical laws or to solve some difficult mathematical problems because over a short interval of time or space these laws tend to be approximately simple proportions of the form dQ = f(z)dz. Such problems will be explored in the next chapter and especially in the next calculus course.

To extend the formula dQ = f(z)dz to larger regions of space or time, we introduce the process called ***integration*** in the Chapter 5. In that process it is often necessary to start with differentials of functions. Let us get fluent at calculating differentials.

Example 4.4.1 Finding differentials.
In each of the following, find the differential dy.

1. $y = \sin x$ 2. $y = e^x(x^2+2)$ 3. $y = \sqrt{x^2+3x-1}$

SOLUTION

1. $y = \sin x$: As $f(x) = \sin x, f'(x) = \cos x$. Thus
$$dy = \cos(x)dx.$$
2. $y = e^x(x^2+2)$: Let $f(x) = e^x(x^2+2)$. We need $f'(x)$, requiring the Product Rule.
   We have $f'(x) = e^x(x^2+2) + 2xe^x$, so
$$dy = \big(e^x(x^2+2) + 2xe^x\big)dx.$$
3. $y = \sqrt{x^2+3x-1}$: Let $f(x) = \sqrt{x^2+3x-1}$; we need $f'(x)$, requiring the Chain Rule.
   We have $f'(x) = \frac{1}{2}(x^2+3x-1)^{-\frac{1}{2}}(2x+3) = \frac{2x+3}{2\sqrt{x^2+3x-1}}$. Thus
$$dy = \frac{(2x+3)dx}{2\sqrt{x^2+3x-1}}.$$

Finding the differential dy of $y = f(x)$ i s really no harder than finding the derivative of f; we just *multiply* $f'(x)$ by dx. It is important to remember that we are not simply adding the symbol "dx" at the end.

We have seen a practical use of differentials as they offer a good method of making certain approximations. Another use is *error propagation.* Suppose a length is measured to be x, although the actual value is $x + \Delta x$ (where Δx is the error, which we hope is small). This measurement of x may be used to compute some other value; we can think of this latter value as $f(x)$ for some function f. As the true length is $x + \Delta x$, one really should have computed $f(x + \Delta x)$. The difference between $f(x)$ and $f(x + \Delta x)$ is the propagated error.

Error Analysis We can approximate the propagated error using differentials.

Example 4.4.2 Using differentials to approximate propagated error

A steel ball bearing is to be manufactured with a diameter of 2cm. The manu-facturing process has a tolerance of ± 0.1mm in the diameter. Given that the density of steel is about 7.85g/cm^3, estimate the propagated error in the mass of the ball bearing.

SOLUTION The mass of a ball bearing is found using the equation "mass = volume $\times$ density." In this situation the mass function is a product of the radius of the ball bearing, hence it is $m = 7.85\frac{4}{3}\pi r^3$. The differential of the mass is

$$dm = 31.4\pi r^2 dr.$$

The radius is to be 1cm; the manufacturing tolerance in the radius is ± 0.05mm, or ± 0.005cm. The propagated error is approximately:

$$\begin{aligned} \Delta m &\longrightarrow dm \\ &= 31.4\pi(1)^2(\pm 0.005) \\ &= \pm 0.493\text{g} \end{aligned}$$

Is this error significant? It certainly depends on the application, but we can get an idea by computing the *relative error*. The ratio between amount of error to the total mass is

$$\begin{aligned} \frac{dm}{m} &= \pm\frac{0.493}{7.85\frac{4}{3}\pi} \\ &= \pm\frac{0.493}{32.88} \\ &= \pm 0.015, \end{aligned}$$

or $\pm 1.5\%$.

We leave it to the reader to confirm this, but if the diameter of the ball was supposed to be 10cm, the same manufacturing tolerance would give a propagated error in mass of ± 12.33g, which corresponds to a *percent error* of $\pm 0.188\%$. While the amount of error is much greater ($12.33 > 0.493$), the percent error is much lower.

We first learned of the derivative in the context of instantaneous rates of change and slopes of tangent lines. We furthered our understanding of the power of the derivative by studying how it relates to the graph of a function (leading to ideas of increasing/decreasing and concavity). This chapter has put the derivative to yet more uses:

- Equation solving (Newton's Method),
- Related Rates (furthering our use of the derivative to find instantaneous rates of change),
- Optimization (applied extreme values), and
- Differentials (useful for various approximations and for something called integration).

In the next chapters, we will consider the "reverse" problem to computing the derivative: given a function f, can we find a function whose derivative is f?
Being able to do so opens up an incredible world of mathematics and applications.

Exercises 4.4

Terms and Concepts

1. T/F: Given a differentiable function $y = f(x)$, we are generally free to choose a value for dx, which then determines the value of dy.

2. T/F: The symbols "dx" and "Δx" represent the same concept.

3. T/F: The symbols "dy" and "Δy" represent the same concept.

4. T/F: Differentials are important in the study of integration.

5. How are differentials and tangent lines related?

6. T/F: In real life, differentials are used to approximate function values when the function itself is not known.

Problems

In Exercises 7 – 16, use differentials to approximate the given value by hand.

7. 2.05^2

8. 5.93^2

9. 5.1^3

10. 6.8^3

11. $\sqrt{16.5}$

12. $\sqrt{24}$

13. $\sqrt[3]{63}$

14. $\sqrt[3]{8.5}$

15. $\sin 3$

16. $e^{0.1}$

In Exercises 17 – 30, compute the differential *dy*.

17. $y = x^2 + 3x - 5$

18. $y = x^7 - x^5$

19. $y = \dfrac{1}{4x^2}$

20. $y = (2x + \sin x)^2$

21. $y = x^2 e^{3x}$

22. $y = \dfrac{4}{x^4}$

23. $y = \dfrac{2x}{\tan x + 1}$

24. $y = \ln(5x)$

25. $y = e^x \sin x$

26. $y = \cos(\sin x)$

27. $y = \dfrac{x+1}{x+2}$

28. $y = 3^x \ln x$

29. $y = x \ln x - x$

30. $f(x) = \ln\left(\sec x\right)$

Exercises 31 – 34 use differentials to approximate propagated error.

31. A set of plastic spheres are to be made with a diameter of 1cm. If the manufacturing process is accurate to 1mm, what is the propagated error in volume of the spheres?

32. The distance, in feet, a stone drops in t seconds is given by $d(t) = 16t^2$. The depth of a hole is to be approximated by dropping a rock and listening for it to hit the bottom. What is the propagated error if the time measurement is accurate to $2/10^{\text{ths}}$ of a second and the measured time is:

    (a) 2 seconds?

    (b) 5 seconds?

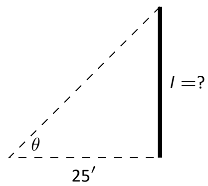
33. What is the propagated error in the measurement of the cross sectional area of a circular log if the diameter is measured at $15''$, accurate to $1/4''$?

34. A wall is to be painted that is $8'$ high and is measured to be $10'$, $7''$ long. Find the propagated error in the measurement of the wall's surface area if the measurement is accurate to $1/2''$.

Exercises 35 – 39 explore some issues related to surveying in which distances are approximated using other measured distances and measured angles. *(Hint: Convert all angles to radians before computing.)*

35. The length l of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured to be 85.2°, accurate to 1°. Assume that the triangle formed is a right triangle.

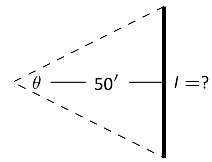
35, 36



(a) What is the measured length l of the wall?
(b) What is the propagated error?
(c) What is the percent error?

36. Answer the questions of Exercise 35, but with a measured angle of 71.5°, accurate to 1°, measured from a point $100'$ from the wall.

37. The length l of a long wall is to be calculated by measuring the angle θ shown in the diagram (not to scale). Assume the formed triangle is an isosceles triangle. The measured angle is 143°, accurate to 1°.



(a) What is the measured length of the wall?
(b) What is the propagated error?
(c) What is the percent error?

38. The length of the walls in Exercises 35 – 37 are essentially the same. Which setup gives the most accurate result?

39. Consider the setup in Exercise 37. This time, assume the angle measurement of 143° is exact but the measured $50'$ from the wall is accurate to $6''$. What is the approximate percent error?

Solutions 4.4

1. T
2. F
3. F
4. T
5. Answers will vary.
6. T
7. Use $y = x^2$; $dy = 2x \cdot dx$ with $x = 2$ and $dx = 0.05$. Thus $dy = .2$; knowing $2^2 = 4$, we have $2.05^2 \approx 4.2$.
8. Use $y = x^2$; $dy = 2x \cdot dx$ with $x = 6$ and $dx = -0.07$. Thus $dy = -0.84$; knowing $6^2 = 36$, we have $5.93^2 \doteq 35.16$.
9. Use $y = x^3$; $dy = 3x^2 \cdot dx$ with $x = 5$ and $dx = 0.1$. Thus $dy = 7.5$; knowing $5^3 = 125$, we have 5.1^3 132.5.
10. Use $y = x^3$; $dy = 3x^2 \cdot dx$ with $x = 7$ and $dx = -0.2$. Thus $dy = -29.4$; knowing $7^3 = 343$, we have $6.8^3 \doteq 313.6$.
11. Use $y = \sqrt{x}$; $dy = 1/(2\sqrt{x}) \cdot dx$ with $x = 16$ and $dx = 0.5$. Thus $dy = .0625$; knowing $\sqrt{16} = 4$, we have $\sqrt{16.5} \doteq 4.0625$.
12. Use $y = \sqrt{x}$; $dy = 1/(2\sqrt{x}) \cdot dx$ with $x = 25$ and $dx = -1$. Thus $dy = -0.1$; knowing $\sqrt{25} = 5$, we have $\sqrt{24} \doteq 4.9$.
13. Use $y = \sqrt[3]{x}$; $dy = 1/(3\sqrt[3]{x^2}) \cdot dx$ with $x = 64$ and $dx = -1$. Thus $dy = -1/48 \doteq 0.0208$; we could use $-1/48 \doteq -1/50 = -0.02$; knowing $\sqrt[3]{64} = 4$, we have $\sqrt[3]{63}$ 3.98.
14. Use $y = \sqrt[3]{x}$; $dy = 1/(3\sqrt[3]{x^2}) \cdot dx$ with $x = 8$ and $dx = 0.5$. Thus $dy = 1/24 \doteq 1/25 = 0.04$; knowing $\sqrt[3]{8} = 2$, we have $\sqrt[3]{8.5} \doteq 2.04$.
15. Use $y = \sin x$; $dy = \cos x \cdot dx$ with $x = \pi$ and $dx \doteq -0.14$. Thus $dy = 0.14$; knowing $\sin \pi = 0$, we have $\sin 3 \doteq 0.14$.
16. Use $y = e^x$; $dy = e^x \cdot dx$ with $x = 0$ and $dx = 0.1$. Thus $dy = 0.1$; knowing $e^0 = 1$, we have $e^{0.1} \doteq 1.1$.
17. $dy = (2x + 3)dx$
18. $dy = (7x^6 - 5x^4)dx$
19. $dy = \frac{-2}{4x^3}dx$
20. $dy = 2(2x + \sin x)(2 + \cos x)dx$
21. $dy = \left(2xe^{3x} + 3x^2e^{3x}\right)dx$
22. $dy = \frac{-16}{x^5}dx$
23. $dy = \frac{2(\tan x+1) - 2x\sec^2 x}{(\tan x+1)^2}dx$
24. $dy = \frac{1}{x}dx$
25. $dy = (e^x \sin x + e^x \cos x)dx$
26. $dy = (-\sin(\sin x)\cos x)dx$
27. $dy = \frac{1}{(x+2)^2}dx$
28. $dy = ((\ln 3)3^x \ln x + \frac{3^x}{x})dx$
29. $dy = (\ln x)dx$
30. $dy = \left(\tan x\right)dx$
31. $dV = \pm 0.157$
32. (a) ± 12.8 feet
    (b) ± 32 feet
33. $\pm 15\pi/8 \doteq \pm 5.89\text{in}^2$
34. $\pm 48\text{in}^2$, or $1/3\text{ft}^2$
35. (a) 297.8 feet
    (b) ± 62.3 ft
    (c) $\pm 20.9\%$
36. (a) 298.8 feet
    (b) ± 17.3 ft
    (c) $\pm 5.8\%$
37. (a) 298.9 feet
    (b) ± 8.67 ft
    $\pm 2.9\%$
38. The isosceles triangle setup works the best with the smallest percent error.
39. 1%

We have spent considerable time considering the derivatives of a function and their applications. In the following chapters, we are going to starting thinking in "the other direction." That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. There are numerous reasons this will prove to be useful: these functions will help us compute area, volume, mass, force, pressure, work, and much more.

4.5 Antiderivatives and Indefinite Integration

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

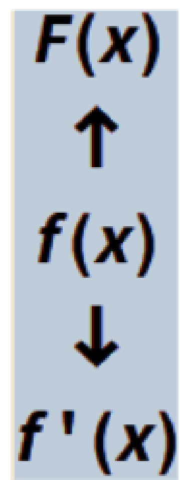
$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.



Definition 4.5.1 Antiderivatives and Indefinite Integrals

Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

The set of all antiderivatives of $f(x)$ is the **indefinite integral of** f, denoted by

$$\int f(x)\, dx.$$

Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them.

We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 4.5.1 Antiderivative Forms

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$ on an interval I. Then there exists a constant C such that, on I,

$$G(x) = F(x) + C.$$

Given a function f defined on an interval I and one of its antiderivatives F, we know *all* antiderivatives of f on I have the form $F(x) + C$ for some constant C. Using Definition 5.1.1, we can say that

$$\int f(x)\ dx = F(x) + C.$$

Let's analyze this indefinite integral notation.

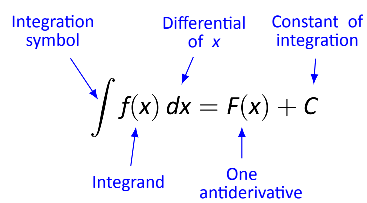


Figure 4.5.1: Understanding the indefinite integral notation.

Figure 5.1.1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Let's practice using this notation.

Example 4.5.1 Evaluating indefinite integrals

Evaluate $\displaystyle\int \sin x\, dx$.

Solution We are asked to find all functions $F(x)$ such that $F'(x) = \sin x$. Some thought will lead us to one solution: $F(x) = -\cos x$, because $\frac{d}{dx}(-\cos x) = \sin x$.

The indefinite integral of $\sin x$ is thus $-\cos x$, plus a constant of integration. So:

$$\int \sin x\, dx = -\cos x + C.$$

A commonly asked question is "What happened to the dx?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin x\, dx$$

presents us with a differential, $dy = \sin x\, dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos x + C$.

Letting $dy = \sin x\, dx$, rewrite

$$\int \sin x\, dx \quad \text{as} \quad \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y + C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos x$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the dx?" with "It went away."

Let's practice once more before stating integration rules.

Example 4.5.2 Evaluating indefinite integrals

Evaluate $\displaystyle\int (3x^2 + 4x + 5)\, dx$.

Solution We seek a function $F(x)$ whose derivative is $3x^2 + 4x + 5$. When taking derivatives, we can consider functions term–by–term, so we can likely do that here.

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int (3x^2 + 4x + 5)\; dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int (3x^2 + 4x + 5)\; dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^3 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$.

This final step of "verifying our answer" is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\; dx\right) = f(x).$$

Differentiation "undoes" the work done by antidifferentiation.

Our derivative tables gave a list of the derivatives of common functions we had learned at that point. We restate part of that list here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn.

Theorem 4.5.2 Derivatives and Antiderivatives

Common Differentiation Rules	Common Indefinite Integral Rules
1. $\frac{d}{dx}(cf(x)) = c\cdot f'(x)$	1. $\int c\cdot f(x)\,dx = c\cdot \int f(x)\,dx$
2. $\frac{d}{dx}\big(f(x)\pm g(x)\big) = f'(x)\pm g'(x)$	2. $\int \big(f(x)\pm g(x)\big)dx = \int f(x)dx \pm \int g(x)dx$
* * * * *	* * * *
3. $\frac{d}{dx}(C) = 0$	3. $\int 0\,dx = C$
4. $\frac{d}{dx}(x) = 1$	4. $\int 1\,dx = \int dx = x + C$
5. $\frac{d}{dx}(x^n) = n\cdot x^{n-1}$	5. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\frac{d}{dx}(\sin x) = \cos x$	6. $\int \cos x\,dx = \sin x + C$
7. $\frac{d}{dx}(\cos x) = -\sin x$	7. $\int \sin x\,dx = -\cos x + C$
8. $\frac{d}{dx}(\tan x) = \sec^2 x$	8. $\int \sec^2 x\,dx = \tan x + C$
9. $\frac{d}{dx}(\csc x) = -\csc x\cot x$	9. $\int \csc x\cot x\,dx = -\csc x + C$
10. $\frac{d}{dx}(\sec x) = \sec x\tan x$	10. $\int \sec x\tan x\,dx = \sec x + C$
11. $\frac{d}{dx}(\cot x) = -\csc^2 x$	11. $\int \csc^2 x\,dx = -\cot x + C$
12. $\frac{d}{dx}(e^x) = e^x$	12. $\int e^x\,dx = e^x + C$
13. $\frac{d}{dx}(a^x) = \ln a\cdot a^x$	13. $\int a^x\,dx = \frac{1}{\ln a}\cdot a^x + C$
14. $\frac{d}{dx}(\ln x) = \frac{1}{x}$	14. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

We highlight a few important points from Theorem 5.1.2:

- Rule #1 states $\int c\cdot f(x)\,dx = c\cdot\int f(x)\,dx$. This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\big(3x^2\big)$ is just as easy to compute as $\frac{d}{dx}\big(x^2\big)$). An example:

$$\int 5\cos x\,dx = 5\cdot\int \cos x\,dx = 5\cdot(\sin x + C) = 5\sin x + C.$$

  In the last step we can consider the constant as also being multiplied by 5, but "5 times a constant" is still a constant, so we just write "C".

- Rule #2 is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example 5.1.2. So:

$$\begin{aligned}\int (3x^2+4x+5)\,dx &= \int 3x^2\,dx + \int 4x\,dx + \int 5\,dx\\ &= 3\int x^2\,dx + 4\int x\,dx + \int 5\,dx\\ &= 3\cdot\frac{1}{3}x^3 + 4\cdot\frac{1}{2}x^2 + 5x + C\\ &= x^3+2x^2+5x+C\end{aligned}$$

  In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- Rule #5 is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n \neq -1$. This is important: $\int \frac{1}{x}\,dx \neq$ "$\frac{1}{0}x^0 + C$"; rather, see Rule #14.
  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     > "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we **first** *multiply* by the power, then **second** *subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: **first** *add* one to the power, then **second** *divide* by the power.

- Note that Rule #14 incorporates the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Initial Value Problems

In Section 2.3 we saw that the derivative of a position function gave a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc. While there is just one derivative of a given function, there are infinitely many antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is -32ft/s^2?", since there is more than one answer.

We can find *the* answer if we provide more information with the question, as done in the following example. Often the additional information comes in the form of an *initial value*, a value of the function that one knows beforehand.

Example 4.5.3 Solving initial value problems

The acceleration due to gravity of a falling object is -32 ft/s^2. At time $t = 3$, a falling object had a velocity of -10 ft/s. Find the equation of the object's velocity.

SOLUTION We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and
- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\, dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$\begin{aligned} v(t) &= -32t + C \\ v(3) &= -10 \\ -32(3) + C &= -10 \\ C &= 86 \end{aligned}$$

Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86$ ft/s. Since the velocity is positive, the object was moving upward.

When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \quad \Rightarrow \quad t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time.

Example 4.5.4 Solving initial value problems

Find $f(t)$, given that $f''(t) = \cos t$, $f'(0) = 3$ and $f(0) = 5$.

SOLUTION We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\, dt = \int \cos t\, dt = \sin t + C = f'(t).$$

So $f'(t) = \sin t + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$f'(0) = 3 \quad \Rightarrow \quad \sin 0 + C = 3 \quad \Rightarrow \quad C = 3.$$

Using the initial value, we have found $f'(t) = \sin t + 3$.

We now find $f(t)$ by integrating again.

$$f(t) = \int f'(t)\,dt = \int (\sin t + 3)\,dt = -\cos t + 3t + C.$$

We are given that $f(0) = 5$, so

$$\begin{aligned} -\cos 0 + 3(0) + C &= 5 \\ -1 + C &= 5 \\ C &= 6 \end{aligned}$$

Thus $f(t) = -\cos t + 3t + 6$.

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a velocity function given an acceleration function.

In the next section, we will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function. Then, in Section 5.4, we will see how areas and antiderivatives are closely tied together. This connection is incredibly important, as indicated by the name of the theorem that describes it: The Fundamental Theorem of Calculus.

The following **memory list** is derived by 'turning around' derivative formulas.

Integral Table

$$\int dx = x + C \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \tan x + C \qquad \int \csc x \cot x\,dx = -\cot x + C$$

Exercises 4.5

Terms and Concepts

1. Define the term "antiderivative" in your own words.

2. Is it more accurate to refer to "the" antiderivative of $f(x)$ or "an" antiderivative of $f(x)$?

3. Use your own words to define the indefinite integral of $f(x)$.

4. Fill in the blanks: "Inverse operations do the ________ things in the ________ order."

5. What is an "initial value problem"?

6. The derivative of a position function is a ________ function.

7. The antiderivative of an acceleration function is a ________ function.

8. If $F(x)$ is an antiderivative of $f(x)$, and $G(x)$ is an antiderivative of $g(x)$, give an antiderivative of $f(x) + g(x)$.

Problems

In Exercises 9 – 27, evaluate the given indefinite integral.

9. $\int 3x^3\,dx$

10. $\int x^8\,dx$

11. $\int (10x^2 - 2)\,dx$

12. $\int dt$

13. $\int 1\,ds$

14. $\int \frac{1}{3t^2}\,dt$

15. $\int \frac{3}{t^2}\,dt$

16. $\int \frac{1}{\sqrt{x}}\,dx$

17. $\int \sec^2\theta\,d\theta$

18. $\int \sin\theta\,d\theta$

19. $\int (\sec x\tan x + \csc x\cot x)\,dx$

20. $\int 5e^\theta\,d\theta$

21. $\int 3^t\,dt$

22. $\int \frac{5^t}{2}\,dt$

23. $\int (2t+3)^2\,dt$

24. $\int (t^2+3)(t^3-2t)\,dt$

25. $\int x^2x^3\,dx$

26. $\int e^\pi\,dx$

27. $\int a\,dx$

28. This problem investigates why Theorem 5.1.2 states that $\int \frac{1}{x}\,dx = \ln|x| + C$.

    (a) What is the domain of $y = \ln x$?

    (b) Find $\frac{d}{dx}\big(\ln x\big)$.

    (c) What is the domain of $y = \ln(-x)$?

    (d) Find $\frac{d}{dx}\big(\ln(-x)\big)$.

    (e) You should find that $1/x$ has two types of antiderivatives, depending on whether $x > 0$ or $x < 0$. In one expression, give a formula for $\int \frac{1}{x}\,dx$ that takes these different domains into account, and explain your answer.

In Exercises 29 – 39, find $f(x)$ described by the given initial value problem.

29. $f'(x) = \sin x$ and $f(0) = 2$

30. $f'(x) = 5e^x$ and $f(0) = 10$

31. $f'(x) = 4x^3 - 3x^2$ and $f(-1) = 9$

32. $f'(x) = \sec^2 x$ and $f(\pi/4) = 5$

33. $f'(x) = 7^x$ and $f(2) = 1$

34. $f''(x) = 5$ and $f'(0) = 7$, $f(0) = 3$

35. $f''(x) = 7x$ and $f'(1) = -1$, $f(1) = 10$

36. $f''(x) = 5e^x$ and $f'(0) = 3, f(0) = 5$

37. $f''(\theta) = \sin\theta$ and $f'(\pi) = 2, f(\pi) = 4$

38. $f''(x) = 24x^2 + 2^x - \cos x$ and $f'(0) = 5, f(0) = 0$

39. $f''(x) = 0$ and $f'(1) = 3, f(1) = 1$

Review

40. Use information gained from the first and second derivatives to sketch $f(x) = \dfrac{1}{e^x + 1}$.

41. Given $y = x^2e^x\cos x$, find dy.

Solutions 4.5

1. Answers will vary.
2. "an"
3. Answers will vary.
4. opposite; opposite
5. Answers will vary.
6. velocity
7. velocity
8. $F(x) + G(x)$
9. $3/4x^4 + C$
10. $1/9x^9 + C$
11. $10/3x^3 - 2x + C$
12. $t + C$
13. $s + C$
14. $-1/(3t) + C$
15. $-3/(t) + C$
16. $2\sqrt{x} + C$
17. $\tan\theta + C$
18. $-\cos\theta + C$
19. $\sec x - \csc x + C$
20. $5e^\theta + C$
21. $3^t/\ln 3 + C$
22. $\frac{5^t}{2\ln 5} + C$
23. $4/3t^3 + 6t^2 + 9t + C$
24. $t^6/6 + t^4/4 - 3t^2 + C$
25. $x^6/6 + C$
26. $e^\pi x + C$
27. $ax + C$
28. (a) $x > 0$
    (b) $1/x$
    (c) $x < 0$
    (d) $1/x$
    (e) $\ln|x| + C$. Explanations will vary.
29. $-\cos x + 3$
30. $5e^x + 5$
31. $x^4 - x^3 + 7$
32. $\tan x + 4$
33. $7^x/\ln 7 + 1 - 49/\ln 7$
34. $5/2x^2 + 7x + 3$
35. $\frac{7x^3}{6} - \frac{9x}{2} + \frac{40}{3}$
36. $5e^x - 2x$
37. $\theta - \sin(\theta) - \pi + 4$
38. $2x^4 + \cos x + \frac{2^x}{(\ln 2)^2} + (5 - \frac{1}{\ln 2})x - 1 - \frac{1}{(\ln 2)^2}$
39. $3x - 2$
40. No answer provided.
41. $dy = (2xe^x\cos x + x^2e^x\cos x - x^2e^x\sin x)dx$

Chapter 5 The Definite Integral

If you know the differential of a quantity $Q = f(t)$, $dQ = r(t)dt$, then by dividing by dt you got

$\frac{dQ}{dt}$ **= r(t)**, the **derivative** (or growth rate).

If you sum up $dQ_i = r(t_i)\,dt$ over a suitable infinite number of t values you will get in this chapter

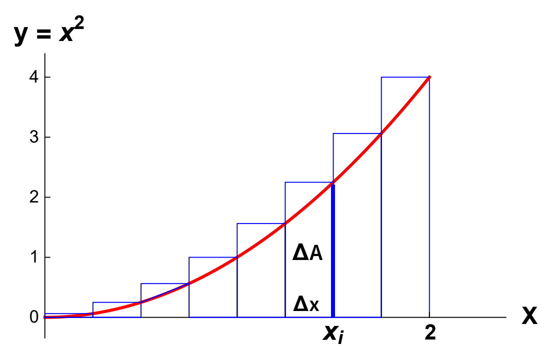
Q(t) - Q(0) = $\int_0^t r(t)\,dt$, the **integral** (or the net change of Q from time 0 to time t).

5.1 We Need (something called) the Definite Integral

Difficult Problems A basic calculus method is to approximate a difficult problem by chopping it up into a large number of ***approximately simple***, easily solved problems. The greater the number of the simple problems, hopefully the better the approximation to the exact answer.

Problem 1 Find the area under the curve $y = f(x) = x^2$ for $0 \le x \le 2$.

Note: $\Delta A_i \doteq f(x_i)\Delta x$. We choose to approximate the area with n rectangles of height $f(x_i)$, right-hand approximation, and width $\Delta x = \frac{2-0}{n}$. Then $x_i = i\,\Delta x = i \cdot \frac{2}{n}$.



The area approximation with n approximating rectangles is then

$$\begin{aligned} A &\doteq \sum_{i=1}^{n} (x_i)^2 \Delta x \\ &= x_1^2 \Delta x + x_2^2 \Delta x + x_3^2 \Delta x + \cdots + x_n^2 \Delta x \\ &= \left(1 \cdot \tfrac{2}{n}\right)^2 \cdot \tfrac{2}{n} + \left(2 \cdot \tfrac{2}{n}\right)^2 \cdot \tfrac{2}{n} + \cdots + \left(n \cdot \tfrac{2}{n}\right)^2 \cdot \tfrac{2}{n} \\ &= \tfrac{8}{n^3}[1^2 + 2^2 + \cdots + n^2] \end{aligned}$$

To compute the table below, you need a good calculator or a Computer Algebra System unless you have lots of time to kill.

Δx	n	A_n
1.	2	5.00000
0.1	20	2.87000
0.01	200	2.68686
0.001	2000	2.66868
0.0001	20000	2.66686
↓	↓	↓
0	+∞	**A = 8/3 ?**

Example Calculation

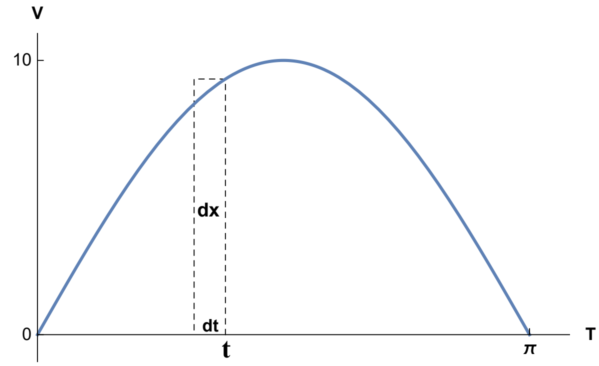
If n = 2,

$$A_2 = \frac{8}{2^3}\,[1^2 + 2^2] = 5$$

Problem 2 The velocity of our charging moose was $v(t) = \frac{dx}{dt} = 10 \sin t \frac{m}{sec}$, $0 \leq t \leq \pi$ seconds.

How close can you safely approach this moose? Use RH approximations.

Note: $\frac{dx}{dt} = v(t) \Rightarrow \Delta x \doteq v(t)\,\Delta t$, **a simple problem**. $\pi = 3.14159\cdots$



Δt	n	x_n
1	3	18.9189
0.1	31	19.9955
0.01	314	19.9999
0.001	3141	19.9999
0.0001	31415	20.0000
↓	↓	↓
0	+∞	**x = 20**

Example Calculation

If $n = 3$.

$x_3 = 10\sin 1 \cdot 1 + 10\sin 2 \cdot 1 + 10\sin 3 \cdot 1$

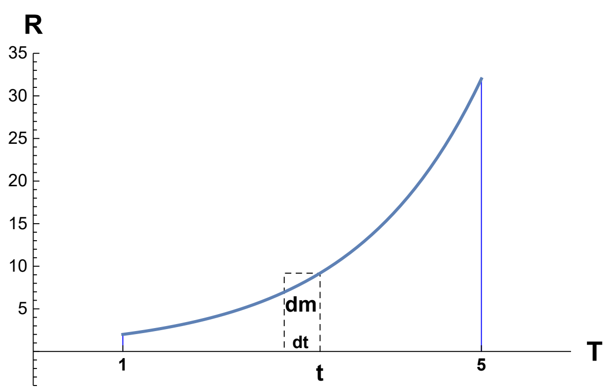
$= 8.4147 + 9.0929 + 1.4112$

$= 18.9189$

Problem 3 The growth rate of a fungus is $r(t) = \frac{dm}{dt} = 2^t \frac{gm}{hour}$, $1 \leq t \leq 5$ hours.

By how much does its mass increase for $1 \leq t \leq 5$ hours? Use RH approximations.

Note: $\frac{dm}{dt} = r(t) \Rightarrow \Delta m \doteq r(t)\Delta t$, **a simple problem.**



Δt	n	m_n
1	4	54.0000
0.1	40	44.7982
0.01	400	43.4310
0.001	4000	43.2959
0.0001	40000	43.2824
↓	↓	↓
0	+∞	**m = 43.2 ⋯**

Example Calculation

If $n = 4$.

$m_4 = 2^2 \cdot 1 + 3^2 \cdot 1 + 4^2 \cdot 1 + 5^2 \cdot 1$

$= 54$

Summary It looks like to find the exact area under a curve $y = f(x)$, we should try

$$A = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

or to recover the size of a quantity whose growth rate is $\frac{dQ}{dt} = r(t)$, try

$$Q = \lim_{n \to +\infty} \sum_{i=1}^{n} r(t_i)\,\Delta t$$

Here is the official name for the above limit of the sum stated in its hyperreal form.

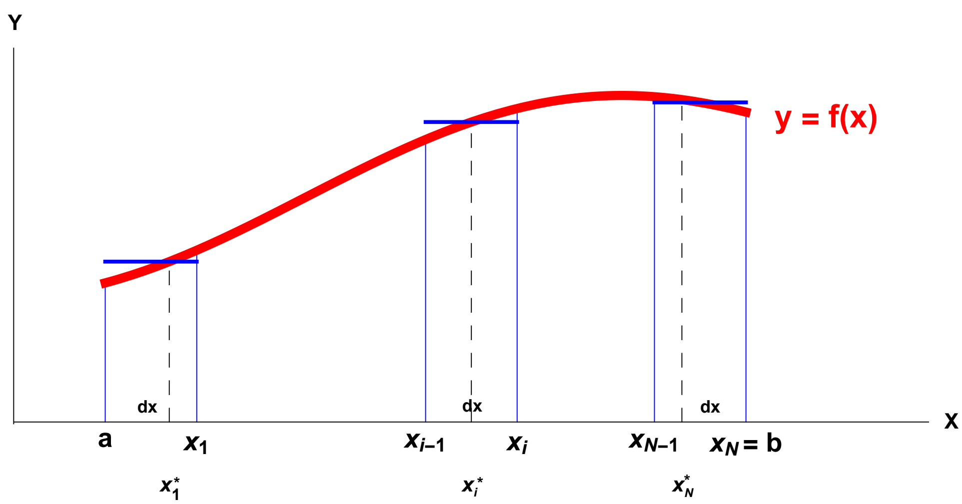
Hyperreal Definition of the (Riemann) Definite Integral

Let f be defined on the interval $a \leqslant x \leqslant b$. Let x_i^* be a point in the i^{th} sub-interval.

Let $dx = \frac{b-a}{N}$, N a positive infinite integer. Then the ***definite integral*** *of f on the interval is*

$$\sum_{i=1}^{N} f(x_i^*)\,dx \approx> \int_a^b f(x)\,dx$$

provided the same result is obtained for any choice of the x_i^*'s.



Note In this definition we require getting the same answer regardless of whether we choose the x_i^*'s to be the left-hand, right-hand, mid-point or any other approximation. ***The most critical step is finding an approximation, while imperfect, gives the exact answer when N is an infinite integer.***

Exercises Show all calculation details.

1. Work #1 with n = 1.
2. Work #2 with n = 3.
3. Work #3 with n = 4.
4. Work #1 with n = 10.
5. Find the area under the curve $y = f(x) = x$ for $0 \le x \le 5$. Use 5 right-hand approximating rectangles. Compare your answer with the exact answer using the formula for the area of a triangle.
6. Find the area under the curve $y = f(x) = x$ for $0 \le x \le 5$. Use 5 left-hand approximating rectangles. Compare your answer with the exact answer using the formula for the area of a triangle.

If you find these calculations interesting, perhaps you should make an appointment with one of our fine school psychologists.

7. Find the area under the curve $y = f(x) = x$ for $0 \le x \le 5$. Use 5 midpoint approximating rectangles.

8. Comment on the results of #5, 6 and 7.

Solutions

1 to 7. For a dollar, I'll do one or two in class or by email upon request, perhaps.

8. #5 gives an over approximation.
#6 gives an under approximation.
#7 gives the exact answer.
Do you understand why?

5.2 The Definite Integral

In the last lecture we learned how to find the area under the curve $y = f(x)$ for $a \le x \le b$ by approximating the area with n rectangles and then expect to get the exact area by letting $n \to +\infty$. It will be helpful in proving theorems about the definite integral to let $n = N$, an infinite integer (the hyperreal approach), and rounding off to get the exact real area.

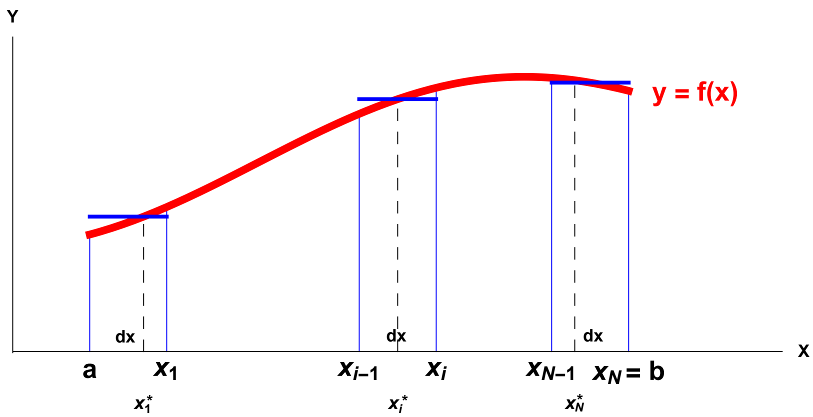
Hyperreal Definition of the (Riemann) Definite Integral

Let f be defined on the interval $a \leqslant x \leqslant b$. Let x_i^* be a point in the i^{th} sub-interval.

Let $dx = \frac{b-a}{N}$, N a positive infinite integer. Then the ***definite integral*** *of f on the interval is*

$$\sum_{i=1}^{N} f(x_i^*)\, dx \approx> \int_a^b f(x)\, dx$$

provided the same result is obtained for any choice of the x_i^*'s.

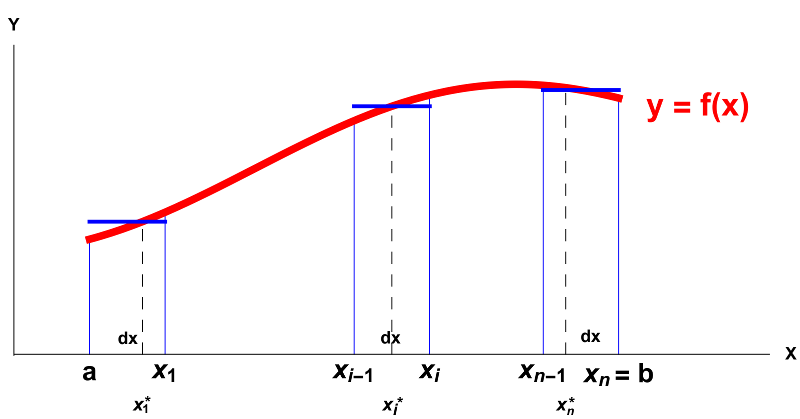


Limit Definition of the (Riemann) Definite Integral

Let f be defined on the interval $a \leqslant x \leqslant b$. Let x_i^* be a point in the i^{th} sub-interval. Let $dx = \frac{b-a}{n}$. Then the definite integral of f on the interval is

$$\int_a^b f(x)\, dx = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i^*)\, dx$$

provided the same result is obtained for any choice of the x_i^*'s.



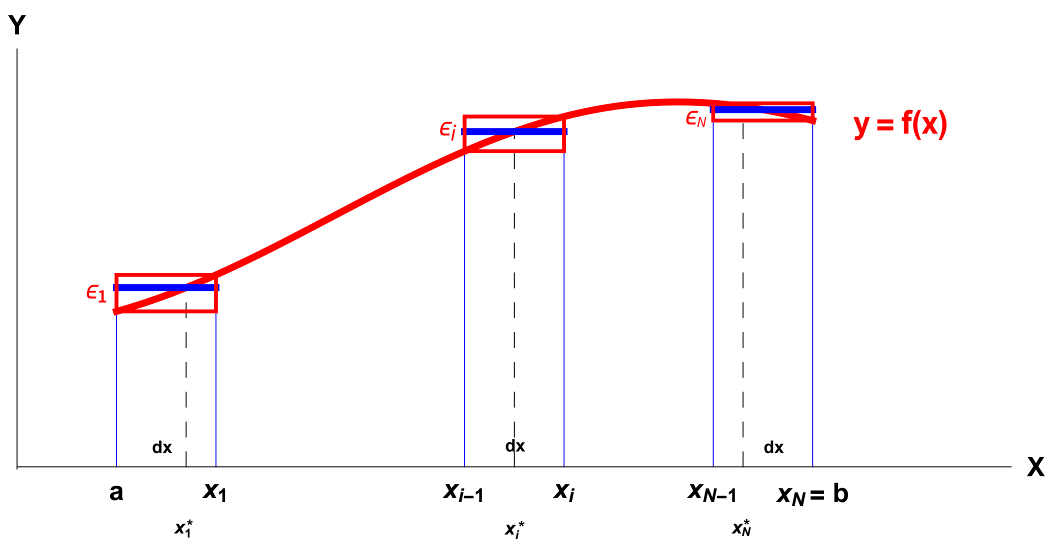
The sum $\sum_{i=1}^{N} f(x_i^*)\,dx$ or $\sum_{i=1}^{n} f(x_i^*)\,dx$ is called a *Riemann Sum*. $\int_a^b f(x)\,dx$ is called a *Riemann Integral*.

Bernhard Riemann, a German mathematician, 1826 to 1866 used these sums. We usually will omit the 'Riemann' because we will only use the Riemann Integral in this course. f is said to be ***Riemann integrable*** on the interval if the integral exits.

For differentiating a function, it must be smooth.
For integrating a function, it is only expected to be continuous.

Integrability Theorem Let f be continuous on the closed interval $a \leqslant x \leqslant b$.
Then f is Riemann integrable over the interval.

Proof



Proof

By the continuity of f, the error rectangles for any choice of the x_i^* all have an infinitesimal height $\leq \epsilon_i$. Let ϵ be the largest of these heights; it also is an infinitesimal. Then the error in calculating $\sum_{i=1}^{N} f(x_i^*)\mathrm{dx}$ is less than or equal to

$\sum_{i=1}^{N} \epsilon_i\,dx$

$\leq \sum_{i=1}^{N} \epsilon\,dx$

$= \epsilon \sum_{i=1}^{N} dx$

$= \epsilon(b-a)$ type i·h

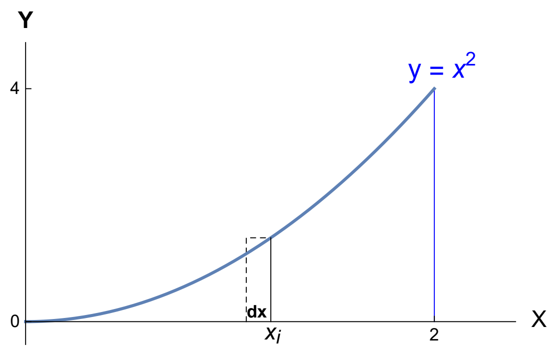
$\approx> 0.$ **End of Proof**

Corollary If f is integrable, then any choice of the x_i^*s can be used to evaluate $\int_a^b f(x)\,dx$.

Example Let's do that for finding the area under the curve $y = f(x) = x^2$ for $0 \leq x \leq 2$ taking $x_i^* = x_i$, the right-hand end point. We got $A \doteq \sum_{i=1}^{n} (x_i)^2 \Delta x = \frac{8}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2]$ for n subdivisions numerically.

You should have learned sigma notation in high school. We will use it sparingly now and treat it in more detail later in calculus. We will use the sum formula

$\sum_{i=1}^{n} i^2 = \frac{1}{6} N(N+1)(2N+1)$ to do this example from the last section exactly using the hyperreal definition of definite integral.



Subdivide the interval $0 \le x \le 2$ into $n = N$, an infinite whole number of subdivisions.

Note: $dx = \frac{2}{N}$ and $x_i = i\,dx = i\frac{2}{N}$. Then

$$\sum_{i=1}^{N} (x_i^*)^2\,dx = \sum_{i=1}^{N} x_i^2\,dx = \sum_{i=1}^{N} \left(i\,\frac{2}{N}\right)^2 \frac{2}{N} = \frac{2^3}{N^3} \sum_{i=1}^{N} i^2 = \frac{2^3}{N^3}\cdot\frac{1}{6} N(N+1)(2N+1)$$

$$= \frac{2^3}{6}\,\frac{N}{N}\,\frac{N+1}{N}\,\frac{2N+1}{N}$$

$$\approx> \frac{2^3}{6}\cdot 1\cdot 1\cdot 2\ \frac{8}{3} = \int_0^2 x^2\,dx.$$

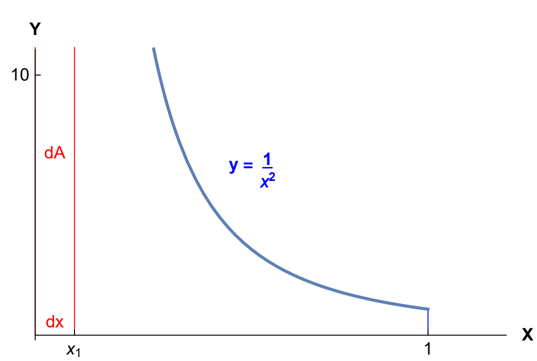
Read it again and agree there has got to be a better way of evaluating integrals exactly!

Outcome Examples The outcomes the hyperreal integral calculation and rounding off can only be:

1. A real number. One example is the problem above. You will see many more of these later.

2. +∞ or -∞. Area under $y = \frac{1}{x^2}$, $0 \le x \le 1$.

The area of the first approximating rectangle alone is $dA = \frac{1}{dx^2}\cdot dx = \frac{1}{dx} \approx> +\infty$!



3. DNE. The area under $y = f(x) = \begin{cases} 1/2 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$.

If the x_i^* are chosen rational, the area of the sum of the approximating rectangles is 1/2. If the x_i^* are chosen irrational, their area is 1.

$\Longrightarrow$ The Riemann Integral $\int_0^1 f(x)\,dx$ DNE.

NOTES

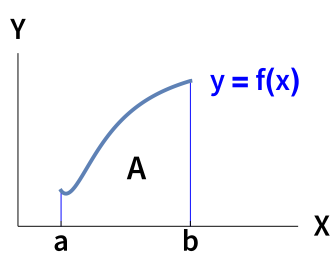
There is a more advanced integral, the Lebesque integral, which evaluates the integral as 1. That will make sense because there are many more irrational numbers than rational ones.

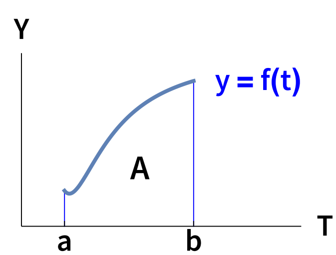
For the rest of this course we will mostly deal with continuous functions. In the next course, in the topic of *generalized functions*, you will learn about discontinuous function calculus.

Let f be continuous. Then by the Integrability Theorem we can choose $x_i^* = x_i$. Then the definition of Definition Integral is much easier to use. But it's still a pain. The really easy way to *integrate* (evaluate the definite integral of f) is found in the Fundamental Theorem of Calculus which we will do in section 4.

Dummy Variable The variable of integration is a ***dummy variable***: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

Note: The word ***dummy*** has largely fallen in disuse. In this context, however, it is an appropriate adjective. Dumb is an Old English word for 'cannot speak' (because of a hearing impairment) which in this context implies 'it does not matter what you call it'*. See the graphs below.





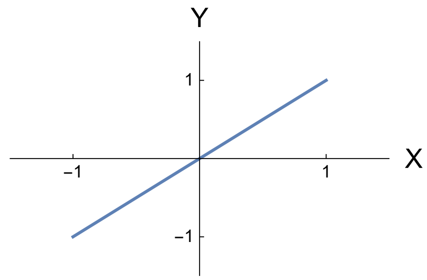
*What do you call a dog without any legs? You don't bother, it won't come anyway.

$\int_a^b f(x)\,dx$ as live math It is often thought that $\int_a^b f(x)\,dx$ is just the name or symbol for $\sum_{i=1}^{N} f(x_i^*)\,dx$ rounded off. If f is continuous on a closed interval $a \leqslant x \leqslant b$, f(x) and dx as hyperreal quantities can be substituted for in $\int_a^b f(x)\,dx$; so it is ***live math***. So you can usually use $\int_a^b f(x)\,dx$ instead of the uglier $\sum_{i=1}^{N} f(x_i^*)\,dx$ in both theory and applications.

Compare this with the derivative situation for $y = f(x)$: $\frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx} \approx> f'(x)$. $\frac{dy}{dx}$ is a complete, concise summary of $\frac{f(x+dx) - f(x)}{dx}$ which $\approx> f'(x)$; that is $\frac{dy}{dx}$ is ***live math***, only infinitesimally wrong, and not just a name for the derivative.

Algebraic or Signed Area In most applications of integration, the integrand can be positive or negative. But if f(x) is negative, so is the area. For uniformity we will allow ourselves to talk about algebraic (signed) area.

Example Find the algebraic area for $f(x) = x$ for $-1 \le x \le 1$. Use the area interpretation.



Algebraic area: $A = -\frac{1}{2} + \frac{1}{2} = 0$. Geometric area: $A = \frac{1}{2} + \frac{1}{2} = 1$.

MidPoint Approximation for $\int_a^b f(x)\,dx$. Take n large and finite (i.e., Δx small) and x_i^* to be the midpoint in the definition of the definite integral.

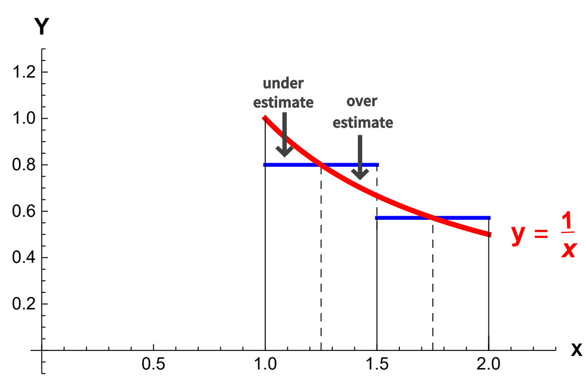
There are many ways of approximating a Riemann integral. The **midpoint approximation** is especially effective because the ‘error triangles' often nearly cancel if there is local linearity.

Example Evaluate $\int_1^2 \frac{1}{x}dx$. Use the mid-point approximation with $n = 2$.

$$\int_1^2 \frac{1}{x}\,dx \doteq \frac{1}{5/4}\cdot\frac{1}{2} + \frac{1}{7/4}\cdot\frac{1}{2} \doteq 0.686$$

Exact value $\doteq$ 0.693

Not bad!



$i = 1$

Exercises

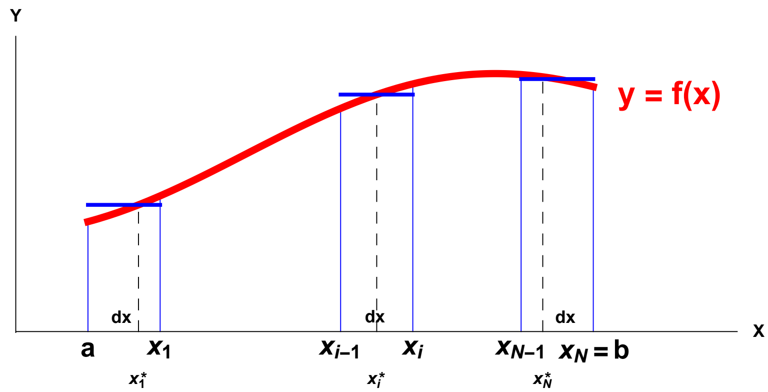
1. Read this section carefully several times. Make sure you understand everything.

2. Evaluate $\int_0^4 \sqrt{x}\,dx$ using the midpoint formula with two subdivisions. Graph. Would you expect your answer to be somewhat, fairly or extremely accurate?

3. Find the area under the curve $y = f(x) = x^3$ for $0 \le x \le 2$ taking $x_i^* = x_i$, the right hand end point.
   Hint: use the formula $\sum_{i=1}^{N} i^3 = \frac{N^2(N+1)^2}{4}$.

5.3 Properties of the Definite Integral

REVIEW Hyperreal Definition of the (Riemann) Definite Integral

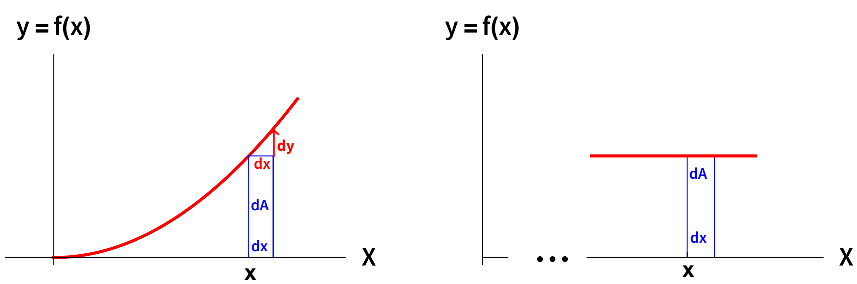


Let f be defined on the interval $a \leqslant x \leqslant b$. Then the ***definite integral*** *of f on the* interval *is*

$$\sum_{i=1}^{N} f(x_i^*)\,dx \approx> \int_a^b f(x)\,dx, \text{ for N a positive infinite integer and } dx = \frac{b-a}{N}$$

provided the same result is obtained for every choice of the x_i^*'s.

The Secret of Integral Calculus Over a short interval a continuous function appears constant.



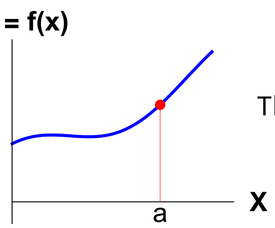
On the left diagram with dx shown infinitely magnified. It appears there may be a significant error in writing $dA = f(x)dx$. On the right diagram the X-axis is infinitely magnified. If f is continuous, then f(x) appears constant (dx an infinitesimal ⇒ dy, an infinitesimal) and so $dA \approx f(x)dx$.

Properties of Integrals **The proofs of most of these are left as exercises; they are completely intuitive in terms of the area interpretation.**

In the definition of Definite Integral, it was assumed that $a < b$. Sometimes it is desirable to remove this restriction.

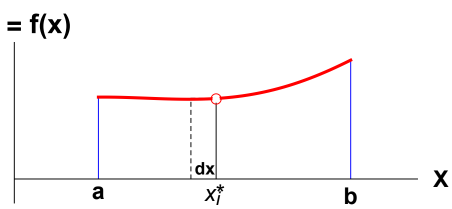
Properties 1 Generalized Limits of Integration (in the definition of definite integral it was assumed that $b > a$, a and b real numbers.

A. $\int_a^a f(x)\,dx = 0$



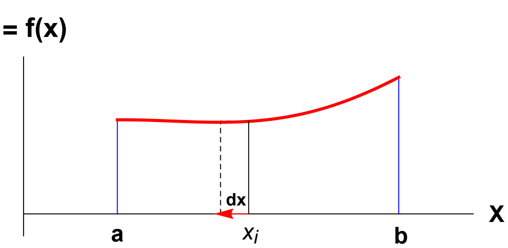
The area under a point is 0.

Generalized Integrals: a related note



In some applications a function may be undefined at a point x_i^* in the interval of integration. Then $f(x_i^*)dx$ is undefined and so $\int_a^b f(x)dx$ is undefined. However, since the area under a point is 0, we will usually ignore this 'undefined infinitesimal area'. With this understanding the integral is defined and is called the '***generalized integral of f on the on the interval a ≤ x ≤ b***.' In applications, this understanding is universally accepted.

B. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$



For $\int_b^a f(x)\,dx$, the dx's in $\sum_{i=1}^{N} f(x_i^*)\,dx$ **are all negative.**

Properties 2. Linearity Properties

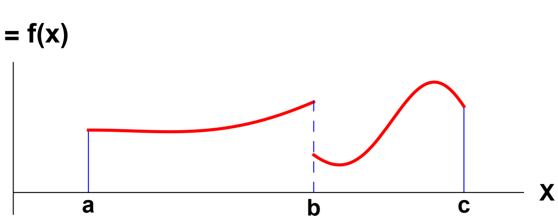
1. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Discussion

1. If you multiply the height of each approximating rectangle by c, you multiply the area of each rectangle by c.
2. If you add the heights of two rectangles, you add their areas.

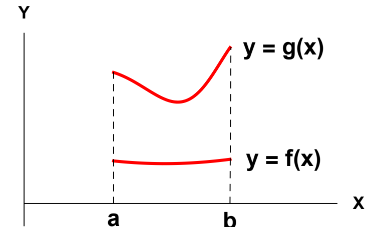
Property 3. Piecewise Continuous Function Property

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



Property 4. Inequality property

$$f(x) \le g(x) \Rightarrow \int_a^b f(x)\,dx \le \int_a^b f(x)\,dx$$



A Historical / Application Note

Historically to find a quantity Q, you begin by finding its differential dQ, because over an infinitesimal time interval, its behavior should be quit simple:

$$dQ \approx r(t)\,dt.$$

To find its rate you divide you divide by dt to get its rate of change, the **derivative**:

$$\frac{dQ}{dt} = r(t).$$

To find its total amount of accumulation of Q, you sum the differential over the time interval obtaining its **integral**:

$$\Delta Q = \int_{t_1}^{t_2} r(t)\,dt$$

There you go. Just about all you need to know about elementary calculus!

Details of the last calculation.

$$dQ = r(t)\,dt$$

$\int_{t_1}^{t_2} dQ = \int_{t_1}^{t_2} r(t)\,dt$ summing / integrating from t_1 to t_2

$Q(t)\big|_{t_1}^{t_2} = \int_{t_1}^{t_2} r(t)\,dt$ or

$Q(t_2) - Q(t_2) = \int_{t_1}^{t_2} r(t)\,dt$ or

$Q_2 - Q_2 = \int_{t_1}^{t_2} r(t)\,dt$ or

$\boxed{\Delta Q = \int_{t_1}^{t_2} r(t)\,dt}$ **The Net Change Theorem (as some call it)**

Think again about this picture. In early calculus, the differential was the key ingredient.

$\Delta Q = \int_{t_1}^{t_2} r(t)\,dt$ **the definite integral**

↑

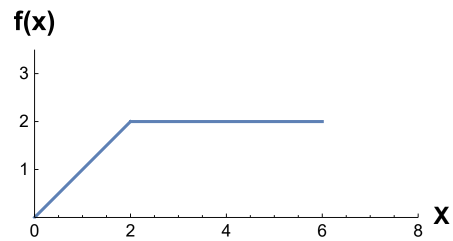
$dQ = r(t)\,dt$ **the differential**

↓

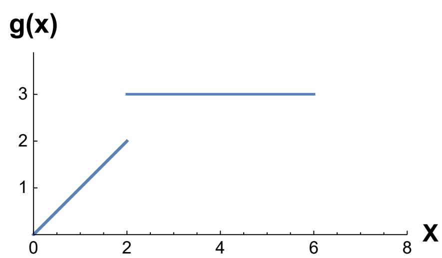
$\frac{dQ}{dt} = r(t)$ **the derivative**

Exercises **Read the section carefully. Semi-memorize the properties of integrals.**

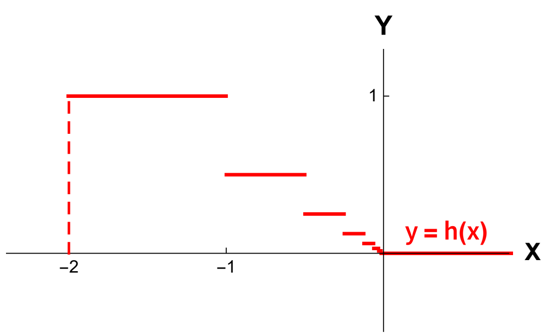
1. Use Property 3 to evaluate $\int_0^6 f(x)\,dx$.



2. Use Property 3 to evaluate the generalized integral $\int_0^6 g(x)\,dx$.



3. Use Property 3 to evaluate $\int_{-2}^0 h(x)\,dx$. Comment: Property 3 sometimes holds even if there are an infinite number of discontinuities.



Hint: A geometric Series. Answer: $A = \frac{4}{3}$.

4. Carefully prove Properties 2 using the definition of definite integral. Illustrate graphically.

5. Carefully prove Property 4 using the definition of definite integral.

Solution

3. $\text{Area} = 1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \cdots$

$= \frac{4}{3}$ Simplifying and using the Geometric Series formula.

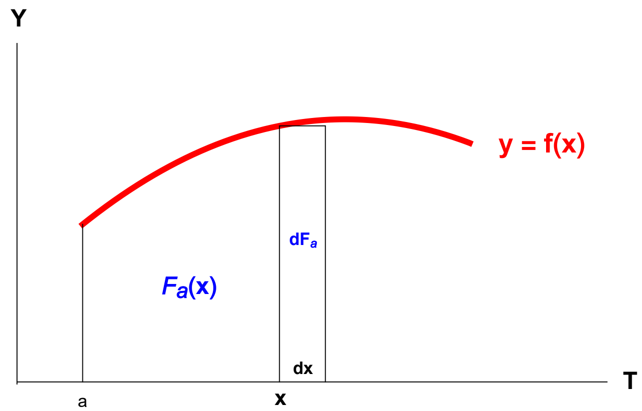
5.4A The Fundamental Theorem of Calculus, I

This is step 1 in finding an easy way to evaluate integrals!

Fundamental Theorem of Calculus, Part I **Let f be continuous for $a \le x \le b$, then**

$$F_a(x) = \int_a^x f(t)\,dt$$

is the antiderivative of f(x) for $a \le x \le b$ satisfying $F_a(a) = 0$.



Proof

Let $F_a(x)$ be the area under the curve $y = f(t)$ for $a \le t \le b$. Then

$$\frac{dF_a}{dx} \approx \frac{f(x)dx}{dx}$$

$$= f(x)$$

So $F_a(x)$ is an antiderivative of f(x).

Also, clearly $F_a(a) = 0$.

End of Proof

Recall, $A \approx B$, A is ***asymptotically equal to*** B means $\frac{A}{B} = 1 + \epsilon$ where ϵ is an infinitesimal.

Alternate Form of FTof C, I $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Example

$$\frac{d}{dx}\int_2^x \sqrt{1+t^2}\,dt = \sqrt{1+x^2}$$

Example

$$\frac{d}{dx}\int_x^3 (t^3+3)\,dt = -\frac{d}{dx}\int_3^x (t^3+3)\,dt = -(x^3+3)$$

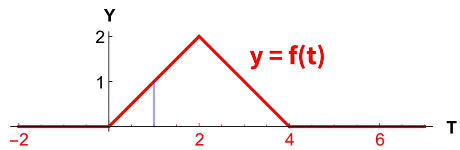
Example

$\frac{d}{dx}\int_0^{x^3} \sin^4 t\,dt$ Hint: Chain Rule. Think $\frac{d}{du}\int_0^u \sin^4 t\,dt$, $u = x^3$

$$= \sin^4 u \cdot 3x^2$$

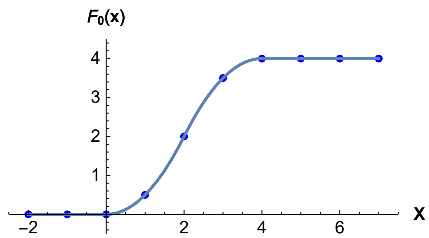
$$= 3x^2 \sin^4 x^3$$

Example* f(x) is the function shown below. Find the antiderivative $F_o(x)$. It is difficult to guess an antiderivative of a piecewise defined function; you will learn how to do this in the next calculus course. We will do it now numerically using the area interpretation, the area between 0 and x. Computers programs can do this accurately even for very complicated functions.



x	-2	-1	0	1	2	3	4	5	6	7
$F_0(x)$	0	0	0	1/2	2	7/2	4	4	4	4

Next, plot the points and join with a smooth curve to get the graph of $F_0(x)$.



NOTE If f(x) is continuous, then $F_a(x)$ is smooth (differentiable). (The proof is an exercise.)

Finally we are almost at the point where evaluating definite integrals is easy. Just let x = b in the Fundamental Theorem of Calculus, I:

$$\int_a^b f(t)\,dt = F_a(b).$$

But we can still do better. It is often easier to find F(x), any antiderivative, than $F_a(x)$.

Exercises

1. Prove that if f(x) is continuous, then $F_a(x)$ is smooth (differentiable).

2. Evaluate

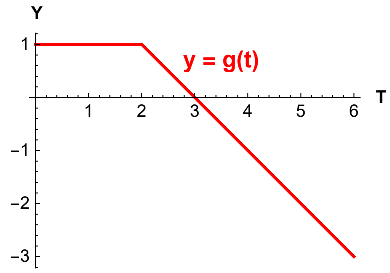
   a. $\frac{d}{dx}\int_3^x \sin^2 t\,dt =$

   b. $\frac{d}{dx}\int_x^5 \sin^2 t\,dt =$

   c. $\frac{d}{dx}\int_{x^2}^5 \sin^2 t\,dt =$

3. Memorize and understand the statement and proof of the **Fundamental Theorem of Calculus, Part I.**

4. Find an antiderivative of $y = g(x)$ graphed below numerically.



Solutions

1. In the proof of the **Fundamental Theorem of Calculus, Part I**, we showed that $F_a(x)$ is differentiable for all x in the interval $a < x < b$. Differentiable means smooth. Remember?

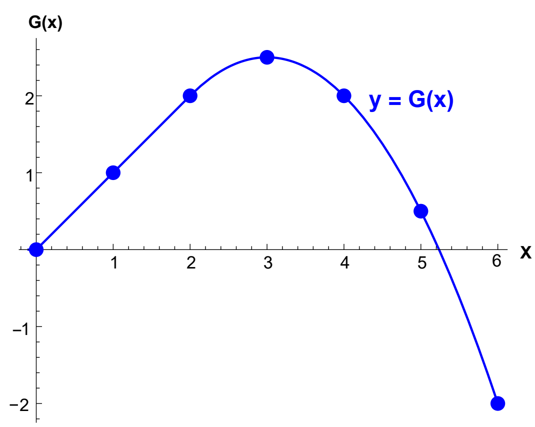
2. **a.** $\sin^2 x$

   b. $-\sin^2 x$

   c. $-2x \sin^2(x^2)$

3. Your job.

4.



5.4 B The Fundamental Theorem of Calculus, Part II

Finally we are almost at the point where evaluating definite integrals is easy. Just let $x = b$ in the Fundamental Theorem of Calculus, I:

$$\int_a^b f(t)\,dt = F_a(b).$$

But we can still do better. It is often easier to find (any) antiderivative $F(x)$ than $F_a(x)$.

Fundamental Theorem of Calculus, Part II Let f be continuous for $a \le t \le b$, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

where $F(x)$ is *any* antiderivative of $f(x)$.

Proof By the FT of C, I

$$\int_a^x f(t)\,dt = F_a(x)$$

The antiderivative satisfying $F_a(a) = 0$

$$= F(x) - F(a)$$

Equivalent form of $F_a(x)$ where $F(x)$ is ***any*** antiderivative of $f(x)$.

Check:

1. $F(x) - F(a)$ is an antiderivative.
2. When $x = a$, $F(x) - F(a) = F(a) = 0$.

Set $x = b$:

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

End of Proof

For easy evaluation, the FTof C, II is usually written

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

which is read “F(x) evaluated between a and b.”

Recall, antiderivative formulas are easier to write and remember in ***indefinite integral*** form.

Recall: Definition $\int f(x)\,dx = F(x) + C$

Example $\int x^3\,dx = \frac{x^4}{4} + C$ is shorter than "If $f(x) = x^3$, then $F(x) = \frac{x^4}{4} + C$."

Application For the quick evaluation of a definite integral, once F(x) is known.

Example $\int_0^4 x^3\,dx = \frac{x^4}{4}\Big|_0^4 = \frac{4^4}{4} - \frac{0^4}{4} = 64$

The following memory list was derived by ‘turning around’ derivative formulas.
In the future you will get more such formulas in a similar way.

Integral Table

$$\int dx = x + C \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

Example Our old friend. Find the area under the curve $y = x^2$ for $0 \le x \le 2$. Finally, the easy way!
Recall $F(x) = \frac{x^3}{3}$. Then

$$A = \int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$$

Example Distance traveled by our moose in section 5.1 with velocity $v = 10 \sin t \ \frac{\text{meters}}{\text{second}}$, $0 \le t \le \pi$. $F(t) = -10 \cos t$. Then

$$x = \int_0^\pi 10 \sin t\,dt = -\cos t\Big|_0^\pi = -10 \cos \pi - (-10 \cos 0) = 10 + 10 = 20 \text{ meter.}$$

Example The fungus problem from Section 5.1. The antiderivative will be found next semester.

$$m = \int_1^5 2^t\,dt = \left.\frac{2^t}{\ln 2}\right|_1^5 = \frac{2^5}{\ln 2} - \frac{2^1}{\ln 2} = \frac{30}{\ln 2} \text{ gram}$$

Example Area under the curve in Example* from 2 to 6.

$$A = \int_2^6 f(x)dx = F(x)\Big|_1^6 = F(6) - F(2) = 4 - 2 = 2.$$

As we remarked at the beginning of this lesson, we have found the easy, almost magical, way of evaluating definite integrals. The problem, though, it is not completely easy to find additional integral formulas. In your next calculus course you will greatly increase the indefinite integral list. However, there will always be many antiderivatives for which you cannot find in a 'nice' form and you may have to evaluate the integrals the 'hard way' (numerically) or ask a Computer Algebra System to do the calculation.

Exercises 5.4 Part B

Terms and Concepts

1. How are definite and indefinite integrals related?

2. What constant of integration is most commonly used when evaluating definite integrals?

3. T/F: If f is a continuous function, then $F(x) = \int_a^x f(t)\, dt$ is also a continuous function.

4. The definite integral can be used to find "the area under a curve." Give two other uses for definite integrals.

Problems

In Exercises 5 – 28, evaluate the definite integral.

5. $\int_1^3 (3x^2 - 2x + 1)\, dx$

6. $\int_0^4 (x-1)^2\, dx$

7. $\int_{-1}^1 (x^3 - x^5)\, dx$

8. $\int_{\pi/2}^{\pi} \cos x\, dx$

9. $\int_0^{\pi/4} \sec^2 x\, dx$

10. $\int_1^e \frac{1}{x}\, dx$

11. $\int_{-1}^1 5^x\, dx$

12. $\int_{-2}^{-1} (4 - 2x^3)\, dx$

13. $\int_0^{\pi} (2\cos x - 2\sin x)\, dx$

14. $\int_1^3 e^x\, dx$

15. $\int_0^4 \sqrt{t}\, dt$

16. $\int_9^{25} \frac{1}{\sqrt{t}}\, dt$

17. $\int_1^8 \sqrt[3]{x}\, dx$

18. $\int_1^2 \frac{1}{x}\, dx$

19. $\int_1^2 \frac{1}{x^2}\, dx$

20. $\int_1^2 \frac{1}{x^3}\, dx$

21. $\int_0^1 x\, dx$

22. $\int_0^1 x^2\, dx$

23. $\int_0^1 x^3\, dx$

24. $\int_0^1 x^{100}\, dx$

25. $\int_{-4}^4 dx$

26. $\int_{-10}^{-5} 3\, dx$

27. $\int_{-2}^2 0\, dx$

28. $\int_{\pi/6}^{\pi/3} \csc x \cot x\, dx$

29. Explain why:

    (a) $\int_{-1}^1 x^n\, dx = 0$, when n is a positive, odd integer, and

    (b) $\int_{-1}^1 x^n\, dx = 2\int_0^1 x^n\, dx$ when n is a positive, even integer.

30. Explain why $\int_a^{a+2\pi} \sin t\, dt = 0$ for all values of a.

In Exercises 31 – 34, do mentally.

31. $\int_0^2 x^2\,dx$

32. $\int_{-2}^2 x^2\,dx$

33. $\int_0^1 e^x\,dx$

34. $\int_0^{16} \sqrt{x}\,dx$

In Exercises 35 – 40, find the average value of the function on the given interval.

35. $f(x) = \sin x$ on $[0, \pi/2]$

36. $y = \sin x$ on $[0, \pi]$

37. $y = x$ on $[0, 4]$

38. $y = x^2$ on $[0, 4]$

39. $y = x^3$ on $[0, 4]$

40. $g(t) = 1/t$ on $[1, e]$

In Exercises 41 – 46, a velocity function of an object moving along a straight line is given. Find the displacement of the object over the given meą interval.

41. $v(t) = -32t + 20$ft/s on $[0, 5]$

42. $v(t) = -32t + 200$ft/s on $[0, 10]$

43. $v(t) = 10$ft/s on $[0, 3]$.

44. $v(t) = 2^t$mph on $[-1, 1]$

45. $v(t) = \cos t$ ft/s on $[0, 3\pi/2]$

46. $v(t) = \sqrt[4]{t}$ ft/s on $[0, 16]$

In Exercises 47 – 50, an acceleration function of an object moving along a straight line is given. Find the change of the object's velocity over the given interval.

47. $a(t) = -32$ft/s^2 on $[0, 2]$

48. $a(t) = 10$ft/s^2 on $[0, 5]$

49. $a(t) = t$ ft/s^2 on $[0, 2]$

50. $a(t) = \cos t$ ft/s^2 on $[0, \pi]$

In Exercises 51 – 54, sketch the given functions and find the area of the enclosed region.

51. $y = 2x$, $y = 5x$, and $x = 3$.

52. $y = -x + 1$, $y = 3x + 6$, $x = 2$ and $x = -1$.

53. $y = x^2 - 2x + 5$, $y = 5x - 5$.

54. $y = 2x^2 + 2x - 5$, $y = x^2 + 3x + 7$.

In Exercises 55 – 58, find $F'(x)$.

55. $F(x) = \int_2^{x^3+x} \frac{1}{t}\,dt$

56. $F(x) = \int_{x^3}^0 t^3\,dt$

57. $F(x) = \int_x^{x^2} (t + 2)\,dt$

58. $F(x) = \int_{\ln x}^{e^x} \sin t\,dt$

Solutions 5.4 B

1. Chain Rule.
2. T
3. $\frac{1}{8}(x^3-5)^8+C$
4. $\frac{1}{4}(x^2-5x+7)^4+C$
5. $\frac{1}{18}\left(x^2+1\right)^9+C$
6. $\frac{1}{3}(3x^2+7x-1)^6+C$
7. $\frac{1}{2}\ln|2x+7|+C$
8. $\sqrt{2x+3}+C$
9. $\frac{2}{3}(x+3)^{3/2}-6(x+3)^{1/2}+C=\frac{2}{3}(x-6)\sqrt{x+3}+C$
10. $\frac{2}{21}x^{3/2}\left(3x^2-7\right)+C$
11. $2e^{\sqrt{x}}+C$
12. $\frac{2\sqrt{x^5+1}}{5}+C$
13. $-\frac{1}{2x^2}-\frac{1}{x}+C$
14. $\frac{\ln^2(x)}{2}+C$
15. $\frac{\sin^3(x)}{3}+C$
16. $-\frac{\cos^4(x)}{4}+C$
17. $-\frac{1}{6}\sin(3-6x)+C$
18. $-\tan(4-x)+C$
19. $\frac{1}{2}\ln|\sec(2x)+\tan(2x)|+C$
20. $\frac{\tan^3(x)}{3}+C$
21. $\frac{\sin\left(x^2\right)}{2}+C$
22. $\tan(x)-x+C$

17. $45/4$
18. $\ln 2$
19. $1/2$
20. $3/8$
21. $1/2$
22. $1/3$
23. $1/4$
24. $1/101$
25. 8
26. 15
27. 0
28. $2-2/\sqrt{3}$
29. Explanations will vary. A sketch will help.
30. $\int_a^{a+2\pi}\sin t\,dt=\cos(a+2\pi)-\cos(a)$. Since cosine is periodic with period 2π, $\cos(a+2\pi)=\cos(a)$, and hence the integral is 0.
31. $c=8/3$
32. $c=16/3$
33. 128/3
34. $2/\pi$
35. $2/pi$
36. 2
37. $16/3$
38. 16
39. $1/(e-1)$
40. -300ft
41. 400ft

43. 30ft
44. $1.5/\ln(2)\doteq 2.164$miles
45. -1ft
46. $128/5$ft
47. -64ft/s
48. 50ft/s
49. 2ft/s
50. 0ft/s
51. $27/2$
52. 21
53. $9/2$
54. $343/6$
55. $F'(x)=(3x^2+1)\frac{1}{x^3+x}$
56. $F'(x)=-3x^{11}$
57. $F'(x)=2x(x^2+2)-(x+2)$
58. $F'(x)=e^x\sin(e^x)-1/x\sin(\ln x)$

5.5 A The Method of Substitution, Indefinite Integrals

In applications, simple integrals like $\int \cos x \, dx$ are rare. It is more likely you will encounter integrals like $\int \cos(2\pi k)\, dx$ or $\int \cos(2.34\, x + 7.49)\, dx$. Fortunately these can often be worked with a slightly modified table of integrals.

Integral Table

$$\int dx = x + C \qquad \int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x \, dx = e^x + C \qquad \int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \cos x \, dx = \sin x + C \qquad \int \sin x \, dx = -\cos x + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C \qquad \int \csc x \cot x \, dx = -\csc x + C$$

Integral Table (Change of variable Form)

$$\int du = u + C \qquad \int u^n \, du = \frac{u^{n+1}}{n+1} + C$$

$$\int e^u \, du = e^u + C \qquad \int a^u \, du = \frac{a^u}{\ln a} + C$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int \cos u \, du = \sin u + C \qquad \int \sin u \, du = -\cos u + C$$

$$\int \sec^2 u \, du = \tan u + C \qquad \int \csc^2 u \, du = -\cot u + C$$

$$\int \sec u \tan u \, du = \sec u + C \qquad \int \csc u \cot u \, du = -\csc u + C$$

Method of Substitution

$$\int f(g(x))\, g'(x)\, dx \underset{du = g'(x)\, dx}{\overset{u = g(x)}{=}} \int f(u)\, du$$

$$\int_a^b f(g(x))\, g'(x)\, dx \underset{du = g'(x)\, dx}{\overset{u = g(x)}{=}} \int_{g(a)}^{g(b)} f(u)\, du$$

Proofs: an integral is live mathematics.

Looking Ahead

Examples

$$\int \sin^3 x \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int u^3 \, du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{1}{4} \sin^4 x + C.$$

$$\int_0^4 \sqrt{2x+1} \, dx$$

$u = 2x + 1$

$du = 2dx \implies dx = \frac{du}{2}$

$x = 0 \implies u = 2{\cdot}0 + 1 = 1$

$x = 4 \implies u = 2{\cdot}4 + 1 = 9$

$$= \int_1^9 \sqrt{u} \, \frac{du}{2}$$

$$= \frac{1}{2} \int_1^9 u^{1/2} \, du$$

$$= \frac{1}{2} \, \frac{2}{3} \, u^{3/2} \Big|_1^9$$

$$= \frac{1}{3} (9 - 1)$$

$$= \frac{8}{3}$$

The Method

Choose a u for which there is (up to a constant)

a du in the correct position.

5.5A Substitution Method, Indefinite Integrals, Readings

Example 5.5.1 **Integrating by substitution**

Evaluate $$\int x\sin(x^2+5)\,dx.$$

We see a u and except for a 2, a du.

Solution

Let $u = x^2+5$, hence $du = 2x\,dx$.

$$du = 2x\,dx \quad\Rightarrow\quad \frac{1}{2}du = x\,dx.$$

Eventually you can do these in your head with perhaps a little 'massaging' of the integrand.

$$\begin{aligned}\int x\sin(x^2+5)\,dx &= \int \sin(\underbrace{x^2+5}_{u})\,\underbrace{x\,dx}_{\frac{1}{2}du}\\ &= \int \frac{1}{2}\sin u\,du\\ &= -\frac{1}{2}\cos u + C\\ &= -\frac{1}{2}\cos(x^2+5)+C.\end{aligned}$$

$$\int x\sin(x^2+5)\,dx.$$

Thinking $u = x^2 + 5$, $du = 2x\,dx$

$$= \tfrac{1}{2}\int \sin(x^2+5)(2x\,dx)$$

$$= -\tfrac{1}{2}\cos(x^2+5) + C$$

Example 5.2 **Integrating by substitution**

Evaluate $\int \sin x\cos x\,dx$.

We see a u and exactly, a du.

Solution

In this example, let's set $u = \sin x$. Then $du = \cos x\,dx$, which we have as part of the integrand! The substitution becomes very straightforward:

$$\begin{aligned}\int \sin x\cos x\,dx &= \int u\,du\\ &= \frac{1}{2}u^2 + C\\ &= \frac{1}{2}\sin^2 x + C.\end{aligned}$$

Example 5.5.3 **Integrating by substitution**

$$\int \cos(5x)\,dx.$$

We see a u and except for a 5, a du.

Solution Let $u = 5x$, then $du = 5dx$.

$$\begin{aligned}\int \cos(5x)\,dx &= \int \cos(\underbrace{5x}_{u})\,\underbrace{dx}_{\frac{1}{5}du}\\ &= \int \frac{1}{5}\cos u\,du\\ &= \frac{1}{5}\sin u + C\\ &= \frac{1}{5}\sin(5x)+C.\end{aligned}$$

Exercises 5.5A

Terms and Concepts

1. Substitution "undoes" what derivative rule?

2. T/F: One can use algebra to rewrite the integrand of an integral to make it easier to evaluate.

Problems

In Exercises 3 – 14, evaluate the indefinite integral to develop an understanding of Substitution.

3. $\int 3x^2\left(x^3-5\right)^7\,dx$

4. $\int (2x-5)\left(x^2-5x+7\right)^3\,dx$

5. $\int x\left(x^2+1\right)^8\,dx$

6. $\int (12x+14)\left(3x^2+7x-1\right)^5\,dx$

7. $\int \frac{1}{2x+7}dx$

8. $\int \frac{1}{\sqrt{2x+3}}dx$

9. $\int \frac{x}{\sqrt{x+3}}dx$ Let u = x+3

10. $\int \frac{x^3-x}{\sqrt{x}}dx$ Simplify integrand

11. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

12. $\int \frac{x^4}{\sqrt{x^5+1}}dx$

13. $\int \frac{\frac{1}{x}+1}{x^2}dx$ Simplify integrand

14. $\int \frac{\ln(x)}{x}dx$

In Exercises 15 – 24, use Substitution to evaluate the indefinite integral involving trigonometric functions.

15. $\int \sin^2(x)\cos(x)dx$

16. $\int \cos^3(x)\sin(x)dx$

17. $\int \cos(3-6x)dx$

18. $\int \sec^2(4-x)dx$

19. $\int \sec(2x)dx$ Hint: Wolfram Alpha

20. $\int \tan^2(x)\sec^2(x)dx$

21. $\int x\cos\left(x^2\right)dx$

22. $\int \tan^2 x\,dx$ Hint: **$\tan^2 x = 1 - \sec^2 x$**

Solutions 5.5

1. Chain Rule.
2. T
3. $\frac{1}{8}(x^3-5)^8+C$
4. $\frac{1}{4}(x^2-5x+7)^4+C$
5. $\frac{1}{18}\left(x^2+1\right)^9+C$
6. $\frac{1}{3}(3x^2+7x-1)^6+C$
7. $\frac{1}{2}\ln|2x+7|+C$
8. $\sqrt{2x+3}+C$
9. $\frac{2}{3}(x+3)^{3/2}-6(x+3)^{1/2}+C=\frac{2}{3}(x-6)\sqrt{x+3}+C$
10. $\frac{2}{21}x^{3/2}\left(3x^2-7\right)+C$
11. $2e^{\sqrt{x}}+C$
12. $\frac{2\sqrt{x^5+1}}{5}+C$
13. $-\frac{1}{2x^2}-\frac{1}{x}+C$
14. $\frac{\ln^2(x)}{2}+C$
15. $\frac{\sin^3(x)}{3}+C$
16. $-\frac{\cos^4(x)}{4}+C$
17. $-\frac{1}{6}\sin(3-6x)+C$
18. $-\tan(4-x)+C$
19. $\frac{1}{2}\ln|\sec(2x)+\tan(2x)|+C$
20. $\frac{\tan^3(x)}{3}+C$
21. $\frac{\sin\left(x^2\right)}{2}+C$
22. $\tan(x)-x+C$

5.5B The Method of Substitution, Definite Integrals

$$\int_a^b f(g(x))\, g'(x)\, dx \underset{du = g'(x)\,dx}{\overset{u = g(x)}{=}} \int_{g(a)}^{g(b)} f(u)\, du$$

Proof: the integral is live mathematics.

This section has focused on evaluating indefinite integrals as we are learning a new technique for finding antiderivatives. However, much of the time integration is used in the context of a definite integral. Definite integrals that require substitution can be calculated using the following workflow:

1. Start with a definite integral $\displaystyle\int_a^b f(x)\, dx$ that requires substitution.

2. Ignore the bounds; use substitution to evaluate $\displaystyle\int f(x)\, dx$ and find an antiderivative $F(x)$.

3. Evaluate $F(x)$ at the bounds; that is, evaluate $F(x)\Big|_a^b = F(b) - F(a)$.

This workflow works fine, but substitution offers an alternative that is powerful and amazing (and a little time saving).

At its heart, (using the notation of Theorem 6.1.1) substitution converts integrals of the form $\int F'(g(x))g'(x)\, dx$ into an integral of the form $\int F'(u)\, du$ with the substitution of $u = g(x)$. The following theorem states how the bounds of a definite integral can be changed as the substitution is performed.

Theorem 6.1.4 Substitution with Definite Integrals

Let F and g be differentiable functions, where the range of g is an interval I that is contained in the domain of F. Then

$$\int_a^b F'\big(g(x)\big)g'(x)\, dx = \int_{g(a)}^{g(b)} F'(u)\, du.$$

In effect, Theorem 6.1.4 states that once you convert to integrating with respect to u, you do not need to switch back to evaluating with respect to x. A few examples will help one understand.

Example 6.1.16 Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^2 \cos(3x-1)\, dx$ using Theorem 6.1.4.

We see a u
and except for a · · · .

Solution Observing the composition of functions, let $u = 3x - 1$, hence $du = 3dx$. As $3dx$ does not appear in the integrand, divide the latter equation by 3 to get $du/3 = dx$.

By setting $u = 3x - 1$, we are implicitly stating that $g(x) = 3x - 1$. Theorem 6.1.4 states that the new lower bound is $g(0) = -1$; the new upper bound is

$g(2) = 5$. We now evaluate the definite integral:

$$\begin{aligned}\int_0^2 \cos(3x-1)\,dx &= \int_{-1}^5 \cos u\,\frac{du}{3}\\ &= \frac{1}{3}\sin u\Big|_{-1}^{5}\\ &= \frac{1}{3}\big(\sin 5 - \sin(-1)\big) \doteq -0.039.\end{aligned}$$

Notice how once we converted the integral to be in terms of u, we never went back to using x.

The graphs in Figure 6.1.1 tell more of the story. In (a) the area defined by the original integrand is shaded, whereas in (b) the area defined by the new integrand is shaded. In this particular situation, the areas look very similar; the new region is "shorter" but "wider," giving the same area.

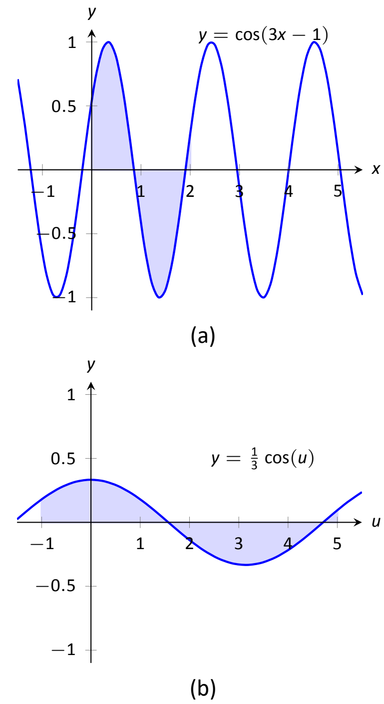


Figure 6.1.1: Graphing the areas defined by the definite integrals of Example 6.1.16.

Example 6.1.17 Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^{\pi/2} \sin x\cos x\,dx$ using Theorem 6.1.4.

Solution We saw the corresponding indefinite integral in Example 6.1.4. In that example we set $u = \sin x$ but stated that we could have let $u = \cos x$. For variety, we do the latter here.

Let $u = g(x) = \cos x$, giving $du = -\sin x\,dx$ and hence $\sin x\,dx = -du$. The new upper bound is $g(\pi/2) = 0$; the new lower bound is $g(0) = 1$. Note how the lower bound is actually larger than the upper bound now. We have

$$\begin{aligned}\int_0^{\pi/2} \sin x\cos x\,dx &= \int_1^0 -u\,du \quad \text{(switch bounds \& change sign)}\\ &= \int_0^1 u\,du\\ &= \frac{1}{2}u^2\Big|_0^1 = 1/2.\end{aligned}$$

In Figure 6.1.2 we have again graphed the two regions defined by our definite integrals. Unlike the previous example, they bear no resemblance to each other. However, Theorem 6.1.4 guarantees that they have the same area.

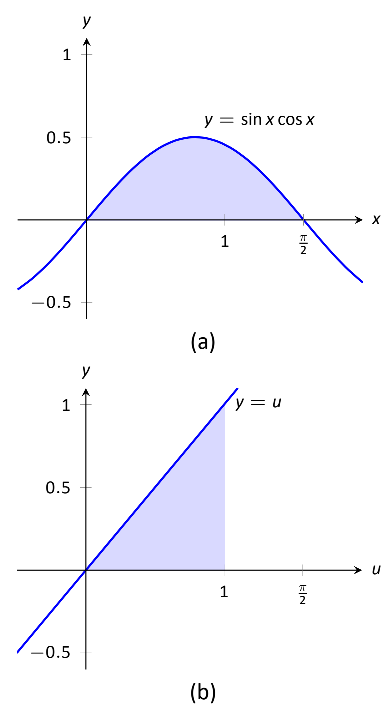


Figure 6.1.2: Graphing the areas defined by the definite integrals of Example 6.1.17.

Exercises 5.5 . Definite Integrals

Evaluate the definite integral.

1. $\displaystyle\int_1^3 \frac{1}{x-5} dx$

2. $\displaystyle\int_2^6 x\sqrt{x-2} dx$

3. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx$

4. $\displaystyle\int_0^1 2x(1-x^2)^4 \, dx$

5. $\displaystyle\int_{-2}^{-1} (x+1)e^{x^2+2x+1} \, dx$

6. $\displaystyle\int_{-1}^1 \frac{1}{1+x^2} \, dx$

7. $\displaystyle\int_2^4 \frac{1}{x^2-6x+10} \, dx$

8. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} \, dx$

Solutions

1. $-\ln 2$
2. $352/15$
3. $2/3$
4. $1/5$
5. $(1-e)/2$
6. $\pi/2$
7. $\pi/2$
8. $\pi/6$

Calculus I introduces you to the main ideas of the calculus.
Calculus II gives you a solid foundation in the calculus and applications and will be able to engage in an intelligent conversation with an engineer.
$MJL@=J ;9D;MDMK ;GMJK=K AFLJG<M;=K QGM LG GL@=J ;9D;MDMKə DAC= L@=GJQ 9F< L@= ;9D;MDMK AF Ґ 9F< ґ <AE=FKAGFKƥ
You can take more advanced calculus based theory and applications courses for the rest of your life.

The calculus is the predominant mathematical tool which was required for the development of our modern industrial and scientific world. While other mathematics topics give us deep insights how the universe works, we would still, with perhaps a little slowdown, be pretty much where we are today with only the calculus.

DONE (Not Really)

If you are an arts student, CONGRATULATIONS.
You deserve a fulfilling life!
If you a serious biology or business student,
take at least one more calculus course.
If you in physical sciences or engineering,
finish the basic calculus sequence and take another math course every term including linear algebra and complex variables.